Simultaneous position and vibration control system for flexible link mechanisms

Paolo Boscariol · Alessandro Gasparetto · Vanni Zanotto

Received: 27 May 2009 / Accepted: 30 June 2010 / Published online: 3 August 2010 © Springer Science+Business Media B.V. 2010

Abstract The control of lightweight and high speed manipulators requires special strategies to prevent and damp the vibrations caused by the inertial components of motion, especially when dealing with flexible-links mechanism. In this paper a constrained MPC (Model Predictive Control) system is proposed as an effective control strategy for position and vibration control of flexible links mechanism. Here this control system is applied to an electric actuated four-link planar mechanism with three flexible links laying on the horizontal plane. The effectiveness of this controller is evaluated by the means of exhaustive numerical simulations.

Keywords FEM analysis · Flexible links · Vibration · Dynamic system · Vibration control

1 Introduction

In the past 40 years modeling, dynamics and control of flexible-links mechanisms have been a central topic in robotics. The fact that accurate dynamic and control of vibration phenomena would allow to design and build robots with reduced weight and higher operative speed has been the main reason of this popularity. Accurate modeling of both single and multi-body flexible links

P. Boscariol (⊠) · A. Gasparetto · V. Zanotto
DIEGM University of Udine, via delle Scienze 208, 33100 Udine, Italy
e-mail: paolo.boscariol@uniud.it

mechanism have been studied in a great deal of works: an extensive review of the results obtained so far can be found in [1]. Among the different approaches to dynamics modeling, Finite Element Method (FEM) is the most popular. This approach, which is based on the discretization of elastic deformation into a finite set of nodal displacements, has been used in [2, 3]. Some authors have also proposed description of flexible mechanisms making use of modal coordinates in places of physical coordinates (see [4, 5]). Other approaches to the dynamic analysis and characterization of planar as well as spatial mechanism can be found in [6, 7]. The vast majority of works has been conducted on singlelink flexible mechanism (see [8-10]) and on multibody systems with only one flexible link, as in [11, 12]. Papers [13] and [14] deal with multibody flexible mechanisms: in the first one Christoforou and Damaren proposes a regulator for controlling a 3 dofs planar manipulator with 2 flexible links, while the latter concern a two planar cooperating 3-link flexible robot with payload. Both linear and nonlinear control strategies have been developed, being the first more frequent, as in [15, 16]. In [16] by Shuchka and Goldenberg an optimal control strategy is applied to control the end-point of a one-link robot. Fung and Chen proposed in [12] a nonlinear controller, while in [17] a neural network is employed by Takahashi and Yamada. Adaptive control is employed with good results in [18, 19]. The latter is also one of the few examples of paper dealing with closed-loop control of a 4-link flexible mechanism.

The aim of this paper is to investigate the effectiveness of a Model-based Predictive Control (MPC) strategy for position and vibration control in a multi-link flexible mechanism, following the results already developed by the same authors in [20]. MPC refers to a family of control algorithms that compute an optimal control sequence based on the knowledge of the plant and on the feedback information. These information, together with a set of constraints, are used as the basis of an optimization problem.

MPC is gaining a wider diffusion in different industrial applications, an interesting report about this specific matter can be found in [21]. This kind of control has been first employed in large chemical factories, but in recent years it has experienced a wider diffusion to other industrial fields. For examples Chen et al. [22] have recently proposed the use of MPC control in ball mill grinding process, while Perez et al. [23] deals with the control of a rudder roll stabilization control for ships. Other interesting results on MPC control of high-bandwidth systems are [24–26].

The availability of more powerful embedded platforms in the last years has encouraged the development of embedded MPC control systems suitable to fast-dynamic plants. For example Hassapis [27] has developed a multicore PC-based embedded MPC control, while FPGA has been chosen by Ling et al. [28] and He et al. [29].

In this paper a model predictive control with constraints is proposed for simultaneous position and vibration control in a four-link flexible mechanism. The choice of this control strategy has been motivated by different factors. First, the prediction ability based on an internal model can be a very effective advantage in a fast-dynamic systems. Then MPC is well suited to MIMO plants (in this case the mechanical system will be modeled as a SIMO plant), since the outputs are computed by solving a minimization problem which can take account of different variables. Another feature of this control strategy is represented by its ability to handle constraints on both control and controlled variables. This can be very effective in realworld control strategies where actuators limitations, such as maximum torque, or maximum speed cannot be neglected. The literature on MPC as an effective vibration reduction strategy in flexible links systems is very limited. To authors' knowledge the only paper focusing on this topic is [30], in which an MPC controller is used to control vibrations in a flexible rotating

beam through electric motor and piezoeceramic actuators. The MPC controller has been implemented in software simulation using Matlab/Simulink. Exhaustive simulations have been made to prove the accuracy and the effectiveness of this control approach. An FPGA implementation of this MPC controller is now being studied, following the results presented by Ling et al. in [28].

The control system proposed in this paper will be employed to control both the position and the vibration in a four-link flexible mechanism laying on the horizontal plane. The crank is actuated by a torque controlled electric motor, while the vibration phenomena are measured in the mid-span of the follower link.

The paper is organized as follows: in Sect. 2 a accurate dynamic model of planar flexible-links mechanism is explained. In Sect. 3 this model is applied to a four-link closed-chain mechanism, and in the same section a linearized model of this mechanism, together with the proof of its accuracy is proposed. In Sect. 4 a brief explanation of MPC control strategy is presented, while in Sect. 5 the numerical results of its application to the reference mechanism are presented.

2 Dynamic model of a four-link planar mechanism

In this section the dynamic model of a flexible-link mechanism proposed by Giovagnoni [3] will be briefly explained. The choice of this formulation among the several proposed in the last 30 years has been motivated mainly by the high grade of accuracy provided by this model, which has been proved several times, for example in [10, 31-35].

The main characteristics of this model can be summarized in four points: 1) finite element (FEM) formulation, 2) Equivalent Rigid-Link System (ERLS) formulation, 3) mutual dependence of rigid and flexible motion, 4) suitability to mechanisms with an arbitrary number of both flexible and rigid links.

First, each flexible link belonging to the mechanism is divided into finite elements. Referring to Fig. 1 the following vectors, calculated in the global reference frame $\{X, Y, Z\}$, can be defined:

- r_i and u_i are the vectors of nodal position and nodal displacement in the *i*th element of the ERLS
- $-\mathbf{p}_i$ is the position of a generic point inside the *i*th element



Fig. 1 Kinematic definitions

q is the vector of generalized coordinates of the ERLS

The vectors defined so far are calculated in the global reference frame $\{X, Y, Z\}$. Applying the principle of virtual work, the following relation can be stated:

$$\sum_{i} \int_{V_{i}} \delta \mathbf{p}_{i}^{T} \ddot{\mathbf{p}}_{i} \rho_{i} dw + \sum_{i} \int_{V_{i}} \delta \epsilon_{i}^{T} \mathbf{D}_{i} \epsilon_{i} dw$$
$$= \sum_{i} \int_{V_{i}} \delta \mathbf{p}_{i}^{T} \mathbf{g} \rho dw + (\delta \mathbf{u}^{T} + \delta \mathbf{r}^{T}) \mathbf{F}$$
(1)

 ϵ_i , \mathbf{D}_i , ρ_i and $\delta\epsilon_i$ are, respectively, the strain vector, the stress-strain matrix, the mass density of the *i*th link and the virtual strains. **F** is the vector of the external forces, including the gravity, whose acceleration vector is **g**. Equation (1) shows the virtual works of, respectively, inertia, elastic an external forces. From this equation, \mathbf{p}_i and $\ddot{\mathbf{p}}_i$ for a generic point in the *i*th element are:

$$\delta \mathbf{p}_{i} = \mathbf{R}_{i} \mathbf{N}_{i} \mathbf{T}_{i} \delta \mathbf{r}_{i}$$

$$\ddot{\mathbf{p}}_{i} = \mathbf{R}_{i} \mathbf{N}_{i} \mathbf{T}_{i} + 2(\dot{\mathbf{R}}_{i} \mathbf{N}_{i} \mathbf{T}_{i} + \mathbf{R}_{i} \mathbf{N}_{i} \dot{\mathbf{T}}_{i}) \dot{\mathbf{u}}_{i}$$
(2)

where \mathbf{T}_i is a matrix that describes the transformation from global-to-local reference frame of the *i*th element, \mathbf{R}_i is the local-to-global rotation matrix and \mathbf{N}_i is the shape function matrix. Taking $\mathbf{B}_i(x_i, y_i, z_i)$ as the strain-displacement matrix, the following relation holds:

$$\epsilon_i = \mathbf{B}_i \mathbf{T}_i \delta \mathbf{u}_i$$

$$\delta \epsilon_i = \mathbf{B}_i \delta \mathbf{T}_i \mathbf{u}_i + \mathbf{B}_i \mathbf{T}_i \delta \mathbf{u}_i$$
(3)

Since nodal elastic virtual displacements ($\delta \mathbf{u}$) and nodal virtual displacements of the ERLS ($\delta \mathbf{r}$) are independent from each other, from the relations reported above the resulting equation describing the motion of the system is:

$$\begin{bmatrix} \mathbf{M} & \mathbf{MS} \\ \mathbf{S}^T \mathbf{M} & \mathbf{S}^T \mathbf{MS} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{S}^T \mathbf{f} \end{bmatrix}$$
(4)

M is the mass matrix of the whole system and **S** is the sensitivity matrix for all the nodes. Vector $\mathbf{f} = \mathbf{f}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{q}, \dot{\mathbf{q}})$ takes account of all the forces affecting the system, including the gravity force. Adding a Rayleigh damping, right-hand side of (4) becomes:

$$\begin{bmatrix} \mathbf{f} \\ \mathbf{S}^T \end{bmatrix} = \begin{bmatrix} -2\mathbf{M}_G - \alpha \mathbf{M} - \beta \mathbf{K} & -\mathbf{M}\dot{\mathbf{S}} & -\mathbf{K} \\ \mathbf{S}^T (-2\mathbf{M}_G - \alpha \mathbf{M}) & -\mathbf{S}^T \mathbf{M} \dot{\mathbf{S}} & 0 \end{bmatrix} \times \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{q}} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{M} & \mathbf{I} \\ \mathbf{S}^T \mathbf{M} & \mathbf{S}^T \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{F} \end{bmatrix}$$
(5)

Matrix \mathbf{M}_G accounts for the Coriolis contribution, while **K** is the stiffness matrix of the whole system. α and β are the two Rayleigh damping coefficients. System in (4) and (5) can be made solvable by forcing to zero as many elastic displacement as the generalized coordinates, in this way ERLS position is defined univocally. So removing the displacement forced to zero from (4) and (5) gives:

$$\begin{bmatrix} \mathbf{M}_{in} & (\mathbf{MS})_{in} \\ (\mathbf{S}^T \mathbf{M})_{in} & \mathbf{S}^T \mathbf{MS} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{in} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{in} \\ \mathbf{S}^T \mathbf{f} \end{bmatrix}$$
(6)

In this way, the values of the accelerations can be computed at each step by solving the system in (6), while the values of velocities and of displacements can be obtained by an appropriate integration scheme (e.g. the Runge-Kutta algorithm). It is important to focus the attention on the size and the rank of the matrices involved, and also to the choice of the general coordinates used in the ERLS definition. Otherwise it might happen that a singular configuration is encountered during the motion of the mechanism. In this case, (6) cannot be solved.

3 Reference mechanism

The mechanism that has been chosen as the basis of the simulations is made by three steel rods. The section of the rods is square, and their side is 6 mm wide.



Fig. 2 The four-link mechanism used for simulations



Fig. 3 Elastic displacements in the four-link mechanism

These three rods are connected on a closed-loop planar chain by three revolute joints. The first and the third link (counting anticlockwise) are connected to a chassis (the fourth link), which can be considered perfectly rigid. The rotational motion of the first link, which is the shortest one, can be imposed through a torquecontrolled electric motor. The whole chain can swing along the horizontal plane, so the effects of gravity on both the rigid and elastic motion of the mechanism can be neglected.

The crank and the coupler have been modeled with a single finite-element. For the follower two finite elements have been used, since it is the longer one. Increasing the number of finite elements will certainly improve the overall accuracy of the model, however

	Symbol	Value
Young's modulus	Ε	210×10^9 [Pa]
Flexural inertia moment	J	$11.102 \times 10^{-10} \text{ [m}^4\text{]}$
Beams width	а	$6 \times 10^{-3} \text{ [m]}$
Beams thickness	b	$6 \times 10^{-3} [m]$
Mass/unit of length of links	т	272×10^{-3} [kg/m]
Crank length	L_1	0.3728 [m]
Coupler length	L_2	0.525 [m]
Follower length	L_3	0.632 [m]
Ground length	L_4	0.3595 [m]
Rayleigh damping constants	α	$8.72 \times 10^{-2} [s^{-1}]$
	β	2.1×10^{-5} [s]

 Table 1
 Kinematic an dynamic characteristics of the flexible link mechanism

this also increases the computational effort required for simulations. Each single finite-element has 6 elastic degrees of freedom, whereas the 2 finite-element has 9 degrees of freedom. The finite elements are Euler-Bernoulli beams. After putting together the 3 links on the frame, and neglecting one of the nodal displacements in order to make the system solvable (see [3]), the resulting flexible system is described by 12 nodal elastic displacements and one rigid degree of freedom. Displacements u_5 and u_9 are measured along the *x*-axis of links 1 and 2, u_6 , u_2 and u_{10} are measured along the *y*-axis, while all the other 7 displacements are angular ones.

3.1 Linearized model

The dynamic model represented by (6) is strongly nonlinear, due to the quadratic relation between the nodal accelerations and the velocities of the free coordinates. Therefore it cannot be used as a prediction model for a linear MPC controller. In order to develop a statespace form linearized version of the dynamic system of (6) a linearization procedure has been developed by Gasparetto [35]. The mentioned procedure will be briefly recalled in this section.

From the basics of system theory, a linear timeinvariant model expressed in state-space can be written as:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{F}_{lin}\mathbf{x}(t) + \mathbf{G}_{lin}\mathbf{w}(t) \\ \mathbf{y}(t) = \mathbf{H}_{lin}\mathbf{x}(t) \end{cases}$$
(7)

where $\mathbf{x}(t)$ is the state vector, $\mathbf{y}(t)$ is the output vector, $\mathbf{w}(t)$ represents the input vector and \mathbf{F}_{lin} , \mathbf{G}_{lin} and \mathbf{H}_{lin} are time-invariant matrices.

From (5) the state vector can be chosen as:

$$\mathbf{x} = [\dot{\mathbf{u}}, \dot{\mathbf{q}}, \mathbf{u}, \mathbf{q}]^T \tag{8}$$

so the linearized state-space form of the dynamic model in (6) can be written as:

$$\mathcal{A}_{lin}\dot{\mathbf{x}} = \mathcal{B}_{lin} \,\mathbf{x} + \mathcal{C}_{lin}\tau \tag{9}$$

Now a steady *equilibrium* configuration \mathbf{x}_e where $\mathbf{u} = \mathbf{u}_e$ under the system input $\mathbf{w} = \mathbf{w}_e$ can be chosen. In the neighborhood of this point the following holds:

$$\begin{cases} \mathbf{x}(t) = \mathbf{x}_e + \Delta \mathbf{x}(t) \\ \mathbf{w}(t) = \mathbf{w}_e + \Delta \mathbf{w}(t) \end{cases}$$
(10)

So, bringing this relations into (9) the following relationship turns out:

$$\mathcal{A}_{lin}(\mathbf{x}_e)\Delta \dot{\mathbf{x}} = \mathcal{B}_{lin}(\mathbf{x}_e + \Delta \mathbf{x})(\mathbf{x}_e + \Delta \mathbf{x}) + \mathcal{C}_{lin}(\mathbf{x}_e + \Delta \mathbf{x})(\mathbf{w}_e + \Delta \mathbf{x})$$
(11)

After some steps that can be found in detail in [35], A_{lin} and B_{lin} in (9) can be written as:

$$\mathcal{A}_{lin} = \begin{bmatrix} \mathbf{M} & \mathbf{MS} & \mathbf{0} & \mathbf{0} \\ \mathbf{S}^{T} \mathbf{M} & \mathbf{S}^{T} \mathbf{MS} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$
(12)
$$\mathcal{B}_{lin} = \begin{bmatrix} -2\mathbf{M}_{G} - \alpha \mathbf{M} - \beta \mathbf{K} & \mathbf{0} & -\mathbf{K} & \mathbf{0} \\ \mathbf{S}^{T} (-2\mathbf{M}_{G} - \alpha \mathbf{M} - \beta \mathbf{K}) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$
(13)

 C_{lin} remains unchanged after the linearization process, since it is composed of only zeros and ones. The standard form of the state-space system can be easily built found from A_{lin} , B_{lin} and C_{lin} :

$$\Delta \dot{\mathbf{x}} = \mathbf{F}_{lin} \Delta \mathbf{x} + \mathbf{G}_{lin} \Delta \mathbf{w}$$

$$\mathbf{y} = \mathbf{H}_{lin} \mathbf{x}$$
 (14)

where:

$$\mathbf{F}_{lin} = \mathcal{A}_{lin}^{-1} \mathcal{B}_{lin}$$

$$\mathbf{G}_{lin} = \mathcal{A}_{lin}^{-1} \mathcal{C}_{lin}$$
(15)

The state vector **x** is measured in the global reference frame so a global-to-local rotation matrix has to be used in order to get output vector **y** with nodal displacements in the local reference frame. Taking q, θ and ϕ as the angular positions of the three flexible links, this conversion matrix can be written as a blockwise diagonal matrix:

$$\mathbf{T}_{LG} = \begin{bmatrix} \mathbf{T}_i(q) & & \\ & \mathbf{T}_i(\theta) & \\ & & \mathbf{T}_i(\phi) \\ & & & \mathbf{T}_i(\phi) \end{bmatrix}$$
(16)

 $\mathbf{T}_i(\phi)$ is employed twice since the follower has been modeled as a double finite element. In oder to get displacement vector **u** measured in the local reference frame and **q** as the output of the state-space model, the following \mathbf{H}_{lin} matrix has to be used:

$$\mathbf{H}_{lin} = \begin{bmatrix} \mathbf{0}^{[12\times12]} & \mathbf{0}^{[12\times1]} & \mathbf{T}_{LG}^{[12\times12]} & \mathbf{1}^{[12\times1]} \end{bmatrix}$$
(17)

3.2 Accuracy of the linearized model

In order to show the accuracy of the linearized model, a comparison based on the impulsive response of the dynamic system will be set. To do this, the mechanism has been fed with a 5 N m torque impulse applied to the crank. The initial configuration has been arbitrarily chosen as $q_0 = \pi/3$, but the effectiveness of the linearization model holds for any configuration of choice. Here only a comparison of the two nodal displacements u_9 and u_{10} is set, however the likeness of the linearized and nonlinear model extends also to all the other 10 nodal displacements belonging to the model.

In Figs. 4 and 5 the evolution of longitudinal and lateral displacements in the midspan of the follower link that occur when the mechanism is fed with an impulsive torque input are shown, while in Fig. 6 the evolution of the mechanism angular position under the same condition is reported. From these graphs, it is clear that the linearized model presents a good level of accuracy, at least as long as the mechanism moves in the neighborhood of its equilibrium configuration. It can be shown that the linearization error on q and the error on u_{10} are less than 0.1% and 20% respectively, as long as the angular position of the first link moves in a ±45 degrees from the linearization configuration. The accuracy of such linear model is inversely proportional to the amplitude of the torque applied to the



Fig. 4 Comparison of the nonlinear vs. linearized system impulsive response: nodal displacement u_9 along the *x*-axis



Fig. 5 Comparison of the nonlinear vs. linearized system impulsive response: nodal displacement u_{10} along the y-axis

crank, however a real mechanism cannot be fed with torque impulses larger than 10 N m without plastic deformations or link breakage. For this reason the proposed comparison addresses to realistic experimental tests.

4 Model Predictive Control with constraints

In this section the equations leading to the constrained MPC system employed will be briefly analyzed. Constrained MPC control is based on these three basic ideas:



Fig. 6 Comparison of the nonlinear vs. linearized system impulsive response: angular position q of the first link

- receding horizon strategy
- internal prediction model
- constraints on both control and controlled variables

In this section a very brief explanation of these concepts is given, for more details see [36].

4.1 Receding horizon strategy

Here a single-input, single-output (SISO) plant will be taken as a matter of example. Defining *k* as the discrete time variable, y(k) and s(k) are the current plant output and the current set-point value respectively, while w(k) is the plant input value. Moreover a reference trajectory r(k|t) can be defined as the ideal trajectory the plant should follow starting from y(k) to reach optimally the set-point trajectory s(k). r(k) can be calculated from the current error $\epsilon(k)$:

$$\epsilon(k) = s(k) - y(k) \tag{18}$$

and $\epsilon(k + 1)$, which is the error found *i* sampling instants later:

$$\epsilon(k+i) = e^{-iT_s/T_{ref}}\epsilon(k) = \lambda^i \epsilon(k) \tag{19}$$

where T_s is the sampling interval and $\lambda = e^{-T_s/T_{ref}} \in (0, 1)$. A suitable formulation for the reference trajectory is:

$$r(k+i|k) = s(k+i) - \epsilon(k+i)$$

= $s(k+i) - e^{-iT_s/T_{ref}}\epsilon(k)$ (20)

where r(k + i|k) is the reference trajectory at time k + i evaluated in k. The availability of an internal prediction model allows to compute an estimation of the future input sequence $\hat{w}(k + i|k)$ with i = 0, 1, ..., $H_p - 1$. H_p is the prediction horizon, i.e. the length measured as a number of discrete time steps, over which an estimation of the plant future dynamic behavior is calculated.

The input sequence $\{\hat{w}(k|k), \hat{w}(k+1|k), \dots, \hat{w}(k+H_p+1|k)\}\$ can be chosen in many different ways. As a first choice, it can be assumed the input remains constant over the prediction horizon: $\hat{w}(k|k) = \hat{w}(k+1|k) = \dots = \hat{w}(k+H_p+1|k).$

After computing the future input sequence, only the first element of this sequence is applied as the input signal to the plant: $w(k) = \hat{w}(k|k)$. At the following sampling interval the sequence of output measurements, predictions and input trajectory calculation is repeated, yielding to: y(k+1), r(k+i|k+1)with $i = 2, 3, \ldots$ The prediction is formulated over k + 1 + i, where $i = 0, 1, ..., H_p - 1$. From those, a new sequence of input values can be calculated: $w(k+1) = \hat{w}(k+1|k+1)$ with $i = 0, 1, \dots, H_p - 1$. Again, only the first element of the reference trajectory is applied to the plant: $w(k+1) = \hat{w}(k+1|k+1)$ and so on. Since the length of the prediction horizon H_p remains constant over the time, and the prediction horizon "slides" forward at each time step, this strategy is commonly mentioned as receding horizon strategy.

4.2 Model prediction

Given a plant model in state-space form:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{w}(k) \\ \mathbf{y}(k) = \mathbf{H}\mathbf{x}(k) \end{cases}$$
(21)

where $\mathbf{x}(k)$ is the state vector, $\mathbf{y}(k)$ and $\mathbf{w}(\mathbf{k})$ are the vectors of outputs and inputs, respectively. **F**, **G** and **H** are the discrete time version of the matrices of LTI linearized model presented in Sect. 3.1. Assuming that the whole state $\mathbf{x}(k)$ is measured, the future behavior of the plant at time k over H_p steps, $[\hat{\mathbf{x}}(k+1|k), \dots, \hat{\mathbf{x}}(k+H_p|k)]$, can be evaluated as:

$$\hat{\mathbf{x}}(k+1|k) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\hat{\mathbf{w}}(k|k)$$
$$\hat{\mathbf{x}}(k+2|k) = \mathbf{F}\hat{\mathbf{x}}(k+1|k) + \mathbf{G}\hat{\mathbf{w}}(k+1|k)$$

÷

$$\hat{\mathbf{x}}(k + H_p|k)$$

$$= \mathbf{F}\hat{\mathbf{x}}(k + H_p - 1|k) + \mathbf{G}\hat{\mathbf{w}}(k + H_p - 1|k)$$

$$= \mathbf{F}^{H_p}\mathbf{x}(k) + \mathbf{F}^{H_p - 1}\mathbf{G}\hat{\mathbf{w}}(k|k) + \cdots$$

$$+ \mathbf{G}\hat{\mathbf{w}}(k + H_p - 1|k)$$
(22)

Following the details reported in [36], last equation can be written in matrix form using the predicted control variable increments $\Delta \hat{\mathbf{w}}$ as:



Prediction values of outputs $(u_9, u_{10} \text{ and } q)$ are calculated from predicted states:

$$\begin{bmatrix} \hat{\mathbf{y}}(k+1|k) \\ \vdots \\ \hat{\mathbf{y}}(k+H_p|k) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{H} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}(k+1|k) \\ \vdots \\ \hat{\mathbf{x}}(k+H_p|k) \end{bmatrix}$$
(24)

4.3 Constrained optimization solution

Supposing to have constraints on both control and controlled variables, and on their change rate, in terms of linear inequalities, such as:

$$w_{i\min} \le w_i(k) \le w_{i\max} \tag{25}$$

 $\Delta w_{i\min} \le \Delta w_i(k) \le \Delta w_{i\max} \tag{26}$

$$z_{i\min} \le z_i(k) \le z_{i\max} \tag{27}$$

These can be expressed as matrix inequalities:

$$\mathbf{V}_1 \begin{bmatrix} \mathcal{W}(k) \\ 1 \end{bmatrix} \le 0 \tag{28}$$

$$\mathbf{V}_2 \begin{bmatrix} \Delta \mathcal{W}(k) \\ 1 \end{bmatrix} \le 0 \tag{29}$$

$$\mathbf{V}_{3} \begin{bmatrix} \mathcal{Z}(k) \\ 1 \end{bmatrix} \le 0 \tag{30}$$

V₁, **V**₂ and **V**₃ are numeric matrices created to establish a matrix expression, while $\mathcal{W} = [\hat{\mathbf{w}}(k|k)^T, \dots, \hat{\mathbf{w}}(k + H_u - 1|k)^T]^T$ is the vector of estimated input values. A similar relation can be used to express also its incremental form $\Delta \mathcal{W}$. $\mathcal{Z}(k)$ instead can be calculated as:

$$\mathcal{Z}(k) = \Psi \hat{\mathbf{x}}(k|k) + \Upsilon \mathbf{w}(k-1) + \Theta \Delta \mathcal{W}(k)$$
(31)

which results from a slightly different matrix rearrangement of (23). Without going into further details, Equations (28)–(30) can be merged in a single inequality:

$$\begin{bmatrix} F \\ \Gamma \Theta \\ \Theta \end{bmatrix} \Delta \mathcal{W}(k) \leq \begin{bmatrix} -F_i \mathbf{w}(k-1) - f \\ -\Gamma \left[\Psi \mathbf{x}(k) + \Upsilon \mathbf{w}(k-1) \right] - g \\ \theta \end{bmatrix}$$
(32)

where F, F_i and f are a subset of V_1 such that $V_1 = [F, \mathbf{f}] = [F_i, \dots, F_{H_p}, \mathbf{f}]$, while V_2 can be split as: $V_2 = [\Gamma, g]$. Θ and θ result from a different formulation of inequality (29), namely:

$$\boldsymbol{\Theta} \Delta \mathcal{W}(k) \le \boldsymbol{\theta} \tag{33}$$

Once all the inequalities constraints are collected in a single formula, as in (32), the focus can be set on the minimization problem, which can be formulated as:

$$\min \mathcal{V}(k) \tag{34}$$

subject to constraints (32), where

$$\mathcal{V}(k) = \sum_{i=1}^{H_p} \|\hat{z}(k+i|k) - r(k+i)\|_{\mathcal{Q}(i)}^2 + \sum_{i=0}^{H_c-1} \|\Delta \hat{w}(k+i|k)\|_{R(i)}^2$$

It can be shown that the minimum of the cost function \mathcal{V} (see [36]) is equal to the minimum of:

$$\min_{\Delta \mathcal{W}(k)} \Delta \mathcal{W}(k)^T \mathcal{H} \Delta \mathcal{W}(k) - \mathcal{G}^T \Delta \mathcal{W}(k)$$
(35)

since $\mathcal{V} = \text{constant} - \Delta \mathcal{W}(k)^T \mathcal{G} + \Delta W(k)^T \mathcal{H} \Delta \mathcal{U}(k)$. Q(i) and R(i) are the *i*th entries of two diagonal matrices of weights. This minimization problem is a standard QP (quadratic programming) problem, since it is in the form: $\min_{\theta} \frac{1}{2} \theta^T \Phi \theta + \phi^T \theta$ with $\Omega \theta \leq \omega$. Moreover, this problem is convex (see [36]), i.e. the local minimum is also the global minimum. Some of the equations shown above contain the state vector **x**, however in practical applications it is impossible to measure all the 12 nodal displacements (and their time derivatives) belonging to the state vector. Hence the need of the state observer to obtain an estimation of the full state vector from a subset of it.

Here a standard Kalman asymptotic estimator has been used. An estimation of $\mathbf{x}(k)$ and $\mathbf{x}_m(k)$ (where $\mathbf{x}(k)$ is the state of the plant model and $\mathbf{x}_m(k)$ is the state of the measurement noise model) can be computed from the measured output $\mathbf{y}(k)$ through:

$$\begin{bmatrix} \hat{\mathbf{x}}(k|k) \\ \hat{\mathbf{x}}_m(k|k) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}(k|k-1) \\ \hat{\mathbf{x}}_m(k|k-1) \end{bmatrix} + \mathcal{M}(\mathbf{y}(k) - \hat{\mathbf{y}}(k))$$
$$\begin{bmatrix} \hat{\mathbf{x}}(k+1|k) \\ \hat{\mathbf{x}}_m(k+1|k) \end{bmatrix} = \begin{bmatrix} \mathbf{F}\hat{\mathbf{x}}(k|k) + \mathbf{F}_u \mathbf{u}(k) \\ \mathbf{F}\hat{\mathbf{x}}_m(k|k) \end{bmatrix}$$
(36)



Fig. 7 Displacement vibration u_{12} : actual value, observed valued and error

 $\hat{\mathbf{y}}(k) = \mathbf{H}\hat{\mathbf{x}}(k|k-1)$

The gain matrix \mathcal{M} is designed using Kalman filtering techniques, see [37]. In this way the state observer can get an accurate estimation of the full state **x** from the knowledge of u_9 , u_{10} and q. A comparison between the observed and the actual displacement u_{12} can be found in Fig. 7 as a basic proof of the capabilities of the state observer.

5 MPC control: numerical results

In this section the results of numerical simulations are provided to show the capabilities of the proposed MPC controller. This controller acts as a MISO (Multiple-Input, Single-Output) system: the MPC relies on the knowledge of the instantaneous values of displacements u_9 and u_{10} and crank angular position q. u_9 , u_{10} and q are the controlled variables, while the torque applied to the crank acts as the control variable. So the tuning of the MPC depends on 6 variables:

1. weight on u_9 : μ_9

- 3. weight on $q: \mu_q$
- 4. sampling time: T_s
- 5. prediction horizon: H_p
- 6. control horizon: H_c

Moreover constraints on both control and controlled variables should to be taken into account. Here the following inequalities constraints have been used:

$u_{9_{min}} \leq u_9 \leq u_{9_{max}};$	$u_{10_{min}} \le u_{10} \le u_{10_{max}}$
$q_{min} \leq q \leq q_{max};$	$\tau_{min} \leq \tau \leq \tau_{max}$

The overall behavior of the controller depends on a large set of variables. While τ_{min} and τ_{max} depend on actuator peak torque, all the others parameters can be tuned quite freely. The weights μ_9 , μ_{10} and μ_q , which belong to the diagonal matrices Q and R in (34) respectively, should be set according to simulation results, since there are no precise rules for an optimal choice. A good balance between the different weights is needed: setting μ_q too high results in a poor vibration damping, while doing the same with μ_9 or μ_{10} produces a very slow and damped response to a step input. Choosing the right tuning of an MPC for a MIMO plant involves another degree of freedom: one or more weights related to controlled variables can be set to zero. In this way it is possible to turn a manipulated variable into an observed one. This can be effective when two controlled variables are strictly coupled, as it happens with nodal displacements u_9 and u_{10} . As a result of all the simulations that led to the tuning of choice, the overall best performance can be obtained setting $\mu_9 = 0$. This choice is also motivated by practical considerations: the magnitude of longitudinal displacement u_9 is so small (around 10^{-4} mm) that can be hardly measured with an adequate accuracy level. Moreover not all the constraints on control and controlled variable must be necessarily used: here the change rate of the control variable is unconstrained.

Other parameters whose values have a strong influence on the closed-loop dynamic behavior are the prediction horizon H_p and the control horizon H_c .

Values of T_s , H_p and H_c should, in practical applications, be chosen according to the available computational resources. Every choice of T_s requires to solve the optimization problem $1/T_s$ times every second, and the computational cost of every evaluation is directly proportional to both H_p and H_c .

As it can be seen from Figs. 8 and 9, MPC can provide a very high vibration damping, in comparison to the performance to the one obtained through a standard PID controller. The tuning of the PID has been done so that the crank is moved at same speed that can be obtained with the MPC controller. All the

^{2.} weight on u_{10} : μ_{10}



Fig. 8 Longitudinal vibration u₉ in the mid-point of the follower link, results obtained with PID, MPC @ 1 kHz, MPC @ 100 Hz



Fig. 9 Transverse vibration u_{10} in the mid-point of the follower, results obtained with PID, MPC @ 1 kHz, MPC @ 100 Hz

tests whose results are displayed here are conducted feeding the controller with a step reference for the angular position of the crank q that goes from 60 degrees to 75 degrees. Comparing the results of the MPC controller with the ones obtained through the PID, it can be seen that the longitudinal displacement u_9 is reduced, while lateral displacement u_{10} is almost eliminated after the mechanism has reached its final position. This MPC implementation shows also a remarkable high-speed in reference following: in more or less 100 ms the mechanism is able to reach its final position showing a very limited overshoot, as can be seen if Fig. 10. This overshoot can be easily eliminated but at the cost of slowing down the reference tracking speed. This can be done by increasing either (or both) the weights μ_{10} and μ_q . The tuning of the MPC is: $T_s = 1$ ms, $H_p = 50$, $H_c = 10$, $\mu_9 = 0$, $\mu_{10} = 1 \times 10^6$, $\mu_q = 1 \times 10^4$. The constraints are set as: $-1 \times 10^{-6} \le u_9 \le 1 \times 10^{-6}$ [m],



Fig. 10 Crank position q, results obtained with PID, MPC @ 1 kHz, MPC @ 100 Hz



Fig. 11 Mechanism configurations: initial, during the motion, final. Elastic deformation is displayed with a $\times 10$ gain

 $-2 \times 10^{-3} \le u_{10} \le 2 \times 10^{-3}$ [m], $\pi/3 \le q \le 1, 4$ [rad], $-10 \le \tau \le 10.$

In order to test a less computational power demanding control system, T_s is set 10 times higher (i.e. $T_s = 10$ ms), and for the same reason both prediction horizon and control horizon are reduced: $H_p = 10$ and $H_c = 5$. All the other parameters are set as in the MPC operating at $T_s = 1$ ms. As it can be seen from Figs. 8 and 9, the system controlled with the MPC with $T_s = 10$ ms shows a less efficient vibration damping, however its performances are still much better than the one obtained through the PID control. This performance degradation in comparison to the higher bandwidth MPC controller can be explained considering that the vibration phenomena the control system has to predict and control are quite fast, therefore small

Table 2 MPC control tuning parameters								
Controller	T_s	H_p	H_c	μ_9	μ_{10}	μ_q		
MPC @ 1 kHz	1 ms	50	10	0	1×10^{6}	1×10^{4}		
MPC @ 100 Hz	10 ms	10	5	0	1×10^{6}	1×10^4		
0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 00	- nomina noisy to - +30% a 0.5	al pruqe and no	isy tor	rque		1.5		

Fig. 12 Robustness analysis to the change of applied torque: angular position q

values of T_s give better results. On the other hand, the reference following, as it can be seen in Fig. 10, is almost not affected by the MPC bandwidth. So, the overall performances of MPC control are very promising even for practical implementations.

In Fig. 11 the mechanism is shown at t = 0, in t = 0.1073 s and in t = 0.5 s, hence respectively before the motion, during the motion (when q reaches $\frac{5}{12}\pi$ for the first time), when the mechanism is at its final configuration. The elastic displacements, calculated using matrix **B**_i, are amplified 10 times in order to make links deformation more evident. As it can be seen in Fig. 11, the largest displacements are located along the crank and the follower.

5.1 Robustness

In Figs. 12 and 13 the effects of gain errors in the estimation of the applied torque are tested. In these figures the results of three tests are displayed in the same graphs: the nominal plant is first controlled with the nominal torque, then a +30% gain error is introduced, finally also an additive white gaussian noise is added to the torque. The MPC controller shows a robust behavior: even when noise is added the response



Fig. 13 Robustness analysis to the change of applied torque: elastic displacement u_{10}

of the system has just a small degradation of the performances.

5.2 Comparison of effective vibration damping

Here a comparison of the damping effects of three control systems is set by comparing the simulated step response of the mechanism using a PID control, an MPC with $T_s = 1$ ms (MPC@1 kHz), and also the MPC with $T_s = 10 \text{ ms}$ (MPC@100 Hz). The tests are conducted using the same tuning parameters, the same initial state of the plant and the same reference signal as the simulations presented in the previous section. As it can be seen in Figs. 14, 15 and 16 the three controllers exhibit different performances. Considering the time required to keep transverse displacement inside ± 1 mm, both MPCs behave considerably better than the PID. The latter requires 350 ms to respect this limit, while the two MPCs need only 16 ms. Then, the PID takes 1.86 s to reduce vibration below 0.1 mm and 2.65 s to get under 0.02 mm, while the MPC with higher T_s requires 0.7 s and 1.8 s respectively. The best performances can be obtained with the MPC working at 1 kHz: 110 ms after the reference step u_{10} is kept below 0.1 mm and after a mere 30 ms below 0.02 mm. The choice of 1 kHz as the sampling frequency of the controller can represent an optimal trade-off between the computational requirements and the performance. Lowering such value reduces the damping performance of the controller, since high frequency modes can lie outside the bandwidth of



Fig. 14 PID control: vibration damping



Fig. 15 MPC control: vibration damping with $T_s = 1$ ms

the observer and the control system, however the stability and the overall performance has been shown to be preserved with T_s as high as 10 ms.

6 Conclusions

A high accuracy FEM-based dynamical model of a four-bar flexible link mechanism is presented in this paper. This model has been employed in software simulation environment to investigate the effectiveness of the MPC control strategy for vibration damping in flexible closed-loop planar mechanisms. In order to implement the control system, a linearized model of the dynamic system has been developed. This linearized state-space model is capable of a high precision approximation of mechanism dynamic behavior, on both position and vibration dynamics. MPC control proved to be very effective both for reference position tracking and vibration suppression. Two implementations of an MPC controller have been tested: a high-bandwidth one and a low-bandwidth one, in order to test the performance of two systems with different computational power needs. The results of both



Fig. 16 MPC control: vibration damping with $T_s = 10$ ms

control systems are compared to the that can be obtained through a standard PID control.

References

- Dwivedy SK, Eberhard P (2006) Dynamic analysis of flexible manipulators, a literature review. Mech Mach Theory 41:749–777
- Nagarajan S, Turcic DA (1990) Lagrangian formulation of the equation of motion for elastic mechanisms with mutual dependence between rigid body and elastic motions. Part II: systems equations. ASME J Dyn Syst Meas Control 112:203–214
- Giovagnoni M (1994) A numerical and experimental analysis of a chain of flexible bodies. J Dyn Syst Meas Control 113:73–80
- Giovagnoni M (1993) Linear decoupled models for a slewing beam undergoing large rotations. J Sound Vib 164:485–501
- Damaren C, Sharf I (1993) Simulation of flexible-link manipulators with inertial and geometric nonlinearities. ASME J Dyn Syst Meas Control 117:74–87
- Di Gregorio R, Parenti Castelli V (2005) On the characterization of the dynamic performances of planar manipulators. Meccanica 40:267–279
- Ali Attia H (2005) Dynamic analysis of spatial linkages: a recursive approach. Meccanica 40:267–279
- Lee AC, Chen ST (1994) Collocated sensor/actuator positioning and feedback design in the control of flexible structure system. ASME J Vib Acoust 116:146–154
- Park JH, Asada H (1994) Dynamic analysis of noncollocated flexible arms and design of torque transmission mechanism. ASME J Dyn Syst Meas Control 116:201–207
- Caracciolo R, Ceresole E, Giovagnoni M (1996) Control experiment of a flexible robot arm using the floating frame model. J Robot Mechatron 8:112–121

- Book WJ, Miaza-Neto O, Witney DE (1975) Feedback control of two beams, two joint system with distributed flexibility. ASME J Dyn Syst Meas Control 97:424–431
- Fung RF, Chen HH (1998) Dynamic analysis and vibration control of a flexible slide-crank mechanism using PM syncronous servo motor drive. J Sound Vib 204:605–637
- Christoforou EG, Damaren CJ (2000) The control of flexible links robots manipulating large payloads. J Robot Syst 17:255–271
- Damaren CJ (2000) On the dynamics and control of flexible multibody systems with closed loops. Int J Robot Res 3:62–75
- Cannon RH, Shmitz E (1984) Initial experiments on the end-point control of a flexible one-link robot. Int J Robot Res 3:62–75
- Shuchka A, Goldenberg AA (1989) Rip control of a singlelink flexible arm using a feedforward technique. Mech Mach Theory 24:439–455
- Takahashi K, Yamada I (1994) Neural-network based learning control of flexible arm using a feedforward technique. ASME J Dyn Syst Meas Control 116:792–795
- Milford RI, Asokatan SF (1994) Experimental on-line frequency domain identification and adaptive control of flexible slewing beam. ASME J Dyn Syst Meas Control 118:56–65
- Trevisani A, Valcher ME An energy-based adaptive control design technique for multibody-mechanisms with flexible links. IEEE/ASME Trans Mechatronics 10:571–580
- Boscariol P, Gasparetto A, Zanotto V (2009) Vibration reduction in a single-link flexible mechanism trough the synthesis of an MPC controller. In: ICM 2009, Malaga, Spain, pp 14–17
- Qin SJ, Baggwell TA (2003) A survey of industrial model predictive control technology. Control Eng Pract 11:733–764
- Chen X-S, Li Q, Fei S-M (2008) Constrained model predictive control in ball mill grinding process. Powder Technol 186:31–39

- Perez T, Goodwin GC, Tzeng CW (2000) Model predictive rudder roll stabilization control for ships. In: 5th IFAC conf on manouvering and control of marine craft, Aalborg, Denmark
- Bleris L, Garcia J, Kothare M (2005) Model predictive hydrodynamic regulations of microflows. In: American control conference, Portland, OR, pp 1752–1757
- Morari M, Baotić M, Borrelli F (2003) Hybrid systems modeling and control. Eur J Control 9(2–3):177–189
- Murray RM, Hauser J, Jadbabie A, Milam MB, Petit N, Dunbar WB, Franz R (2003) Online control customization via optimization-based control. In: Samad T, Balas G (eds) Software-enabled control. IEEE Press/Wiley, New York
- Hassapis G (2003) Implementation of model predictive control using real-time multiprocessing computer. Multiporcess Microsyst 27:327–340
- Ling KV, Yue SP, Maciejowski J (2006) A FPGA implementation of model predictive control. In: American control conference 2006, Minneapolis, pp 1930–1935
- He M, Chen C, Zhang X (2006) FPGA implementation of a recursive rank one updating matrix inversion algorithm for constrained MPC. Intell Control Autom 1:733–737
- Hassan H, Dubay R, Li C, Wang R (2007) Active vibration control of a flexible one-link manipulator using a multivariable predictive controller. Mechatronics 17:311–323

- 31. Gasparetto A, Zanotto V (2006) Vibration reduction in a flexible-link mechanism trough synthesis of an optimal
- Gasparetto A (2004) On the modeling of flexible-link planar mechanisms: experimental validation of an accurate dynamic model. J Dyn Syst Meas Control 126:365–375

controller. Meccanica 41:611–622

- Trevisani A (2003) Feedback control of flexible four-bar linkages: a numerical and experimental investigation. J Sound Vib 268:947–970
- Caracciolo R, Richiedei R, Trevisani A (2006) Design and experimental validation of picewise-linear state observer for flexible link mechanisms. Meccanica 41:623–637
- 35. Gasparetto A (2001) Accurate modeling of a flexible-link planar mechanism by means of a linearized model in the state-space form for design of a vibration control. J Sound Vib 240(2):241–262
- Maciejowski JM (2002) Predictive control with constraints. Prentice Hall, Harlow
- Franklin FG, Powell JD, Workman ML (1990) Digital control of dynamic systems, 2nd edn. Addison Wesley, Reading