# ESDA2012-82103 

# PLANNING CONTUNOUS-JERK TRAJECTORIES FOR INDUSTRIAL MANIPULATORS 

Paolo Boscariol ${ }^{*}$<br>Dipartimento di Ingegneria Elettrica, Gestionale e Meccanica<br>University of Udine<br>Via delle Scienze 208, 33100 Udine Italy<br>Email: paolo.boscariol@uniud.it

Alessandro Gasparetto<br>Dipartimento di Ingegneria Elettrica, Gestionale e Meccanica<br>University of Udine<br>Via delle Scienze 208, 33100 Udine<br>Italy

Renato Vidoni<br>Faculty of Science and Technology<br>Free University of Bozen-Bolzano<br>piazza Università 5,39100 Bolzano<br>Italy


#### Abstract

Planning smooth trajectories is crucial in the most advanced robotic applications in industrial environments. In this paper two novel trajectory planning methods for robotic manipulators are introduced, named " 545 " and " 5455 ". Both methods are based on an interpolation of a sequence of via points using a combination of $4^{\text {th }}$ and $5^{\text {th }}$ order polynomial functions. These techniques allow to obtain a continuous-jerk trajectory for improved smoothness and minimum excitation of vibration. By using the " 545 " method, null jerk at initial time can be achieved, while with the "5455" method one can impose an arbitrary value of jerk at both the first and the last via-point. The outcome of both methods is the optimal time distribution of the via points, with respect to a predefined objective function. Results are provided for a 3 d.o.f. Cartesian manipulator, but the techniques may be applied to any industrial robot.


## INTRODUCTION

The development of trajectory planning algorithms for industrial manipulator is a fundamental topic in robotics engineering. Severe vibrations arise in manipulators when they are moved along a non-smooth trajectory. In that case worsening of accuracy, premature joint wear and mechanical failures might occur. Therefore in the last decades a large number of techniques

[^0]have been developed to design smooth trajectories for industrial robots [1]. Both online and off-line techniques have been developed: this work falls into the latter category. Another fundamental distinction between the several methods available in literature is the use of a model-based or of a model-free approach. While model-based approaches can achieve good results in a specific case [2], they lack the generality which is a fundamental requirement for most industrial application. The development of an accurate dynamic models of a robotic manipulator is a rare practice in most industrial facilities, due to the low level of perceived potential economic advantage and the general lack of the required know-how. Therefore model-free approaches, as the one presented in this paper, are much more appealing for today's market.

In this paper two novel trajectory planning algorithms for industrial robot are presented. Such algorithms produce an optimal trajectory starting from a set of via-points, i.e. they adjust the distance between two consecutive via points in order to minimize a cost function of choice. Constraints on velocity, acceleration and jerk at each joint can be specified as inputs of the optimization procedure. Similar approaches can be encountered quite often in literature, since many methods are available to produce a time law which interpolates or approximate a set of via-points. These works can be classified on the base of different features, such as the choice of the cost function and of the primitive functions used for the interpolation procedure. The paper [3] introduces a method for the evaluation of minimum jerk trajectories as a
global constrained minimax optimization problem. The interpolation is based on a sequence of cubic polynomial functions. Continuous jerk can be achieved by this method, which on the other hand has high computational demands. Literature on trajectory planning has in many cases highlighted the worsening on motion accuracy and the increased level of vibration on the endeffector caused by high level of acceleration and jerk (i.e. the time derivative of acceleration), as in $[4,5]$. In particular Zefran et. al. in [6] shows that smooth trajectories (i.e. trajectories without jerk discontinuities) are to be preferred to comply with the physical limitation of the actuators and with the limits of the control system bandwidth, and that non-smooth motion can excite the structural natural frequencies of the system. A possible solution to the problem of jerk continuity has been solved by the use of fifth order B-spline together with a composite time-jerk cost function are used in [7, 8]. Other popular techniques are based on simpler interpolation functions, such as the " 343 " approach developed by Ho and Cook in [9]. Such method relies on the use of spline functions, but does not guarantee the continuity of jerk along the trajectory. Such problem has been solved by Petrinec and Kovacic in [10], trough the development of the " 445 " trajectory. Such approach makes use of an interpolation function composed by a sequence of $4^{\text {th }}$ order polynomial functions, with a $5^{t h}$ order function for interpolating the motion between the last two via-points. The use of polynomial functions of even high order allows to produce motion profiles with superior smoothness, but at the cost of a general increase of peak values of speed and acceleration, if a constant total execution time is considered. An example of the use of polynomials functions up to the $9^{\text {th }}$ order can be found in [11].

The two innovative methods presented here are based on two new composite trajectory primitives. The first one, named " 545 ", uses a $5^{\text {th }}$ order polynomial function as the primitive for the first and the last segment of the trajectory, while intermediate segments are described by $4^{\text {th }}$ order polynomial functions. This choice, together with a suitable choice of the continuity conditions at each via-point, ensures that:

1. jerk is continuous along the whole trajectory
2. an arbitrary jerk value can be specified for the first point of the trajectory

While the first feature is shared with other algorithms, such as the ones that are based on cubic splines [12], many approaches do not allow for the operator to choose all the kinematic parameters for the first and the last point of the trajectories. As it will be shown in the paper, this feature allows to produce trajectories with null acceleration and jerk at the extreme point of the trajectory. The resulting motion profile is therefore smoother, so it will induce a reduced level of vibration to the robot structure. The second algorithm proposed in this paper is a modified version of " 545 ", named " 5455 ", which allows to impose an arbitrary value of the jerk, as well as acceleration and velocity, at both the
first and the last via point. This feature can be efficiently used to maximize the smoothness of the motion profile during the critical phase of start-up and resting of the robot. Moreover the free choice of initial and final jerk allows to retain maximum smoothness even in the case of the cyclical repetition of the same task, which is case often encountered in industrial applications. Moreover, the two novel methods proposed here are of straightforward implementation. As it will be shown in the paper, the computation of the trajectory requires to find the optimal set of time intervals between two consecutive via points. This is a problem that can be solved quite easily and in a relatively short time by using standard optimization routines, such as Matlab's fmincon. The aforementioned optimization problem is constrained, but its solution is generally not critical, since all the constraints can be expressed as an explicit function of the optimization variable, i.e. the set of time intervals between two via-points.

The evaluation of the effectiveness of the proposed trajectories is conducted by means of extensive experimental results. A three degrees-of-freedom Cartesian manipulator is chosen for testing the novel approach proposed here. Numerical evidences confirm that the proposed method achieves a good level of performance when compared to other popular trajectory planning algorithms [ $3,7,10,13$ ]. The cost function on which the optimization procedure is based is the well known minimum time, which has proved to be a popular choice [14-16], given the appeal of its potential economic advantage for most industrial applications. Here only the case of minimum time with kinematic constraints is considered, but the two novel " 545 " and " 5455 " primitives can be used as a starting point for other innovative methods, simply by changing the choice of the goal function.

## 1 THE "445", " 545 " and " 5455 " TRAJECTORIES

In order to give the clearest explanation possible of the innovative " 545 " trajectory, the " 445 " trajectory is introduced first. The description is made as brief as possible to meet the space constraints of a paper, for a more detailed description please see [10]. Let us consider a trajectory to be planned from $N$ via points $P_{1} \ldots P_{N}$ in the joint space, supposed to be known trough the use of a generic inverse kinematic algorithms from their equivalents in the operative space.

## The " 445 " trajectory

If the " 445 " trajectory is considered, $4^{\text {th }}$ order polynomial function are used to describe all the segments between the via points, with the exception of the last one, for which a $5^{t h}$ order is used. Therefore the trajectory between the two adjacent viapoints $P_{k+1}(1 \leq k \leq N-2)$ can be written as:

$$
\begin{equation*}
F_{k}(t)=B_{k, 1}+B_{k, 2} t+B_{k, 3} t^{2}+B_{k, 4} t^{3}+B_{k, 5} t^{4} \tag{1}
\end{equation*}
$$

in which $F(t)$ represents the position of a joint of the robot under consideration. The boundary conditions for the segment under consideration can be expressed as:

$$
\begin{align*}
& F_{k}(0)=P_{k} \\
& F_{k}\left(T_{k+1}\right)=P_{k+1} \\
& v_{k}(0)=v_{k}  \tag{2}\\
& v_{k}\left(T_{k+1}\right)=v_{k+1} \\
& a_{k}(0)=a_{k}
\end{align*}
$$

in which $v_{k}$ e $v_{k+1}$ are the joint velocities and $a_{k}, a_{k+1}$ are the accelerations at the points $P_{k}$ and $P_{k+1}$, respectively. Using Eqn. 1 in Eqn. 2, one obtains:

$$
\begin{align*}
& F_{k}(0)=B_{k, 1}=P_{k} \\
& v_{k}(0)=B_{k, 2}=v_{k} \\
& a_{k}(0)=2 B_{k, 3}=a_{k} \\
& F_{k}\left(T_{k+1}\right)=B_{k, 1}+B_{k, 2} T_{k+1}+B_{k, 3} T_{k+1}^{2}+ \\
& \quad+B_{k, 4} T_{k+1}^{3}+B_{k, 5} T_{k+1}^{4}=P_{k+1} \\
& v_{k}\left(T_{k+1}\right)=B_{k, 2}+2 B_{k, 3} T_{k+1}+3 B_{k, 4} T_{k+1}^{2}+4 B_{k, 5} T_{k+1}^{3}=v_{k+1} \tag{3}
\end{align*}
$$

Eqn. 3 can be expressed for $1 \leq k \leq N-2$ as:

$$
\left[\begin{array}{c}
B_{k, 1}  \tag{4}\\
B_{k, 2} \\
B_{k, 3} \\
B_{k, 4} \\
B_{k, 5}
\end{array}\right]=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 \\
-\frac{4}{T_{k+1}^{3}} & -\frac{3}{T_{k+1}^{2}} & -\frac{1}{T_{k+1}} & \frac{4}{T_{k+1}^{3}} & -\frac{1}{T_{k+1}^{2}} \\
\frac{3}{T_{k+1}^{4}} & \frac{2}{T_{k+1}^{3}} & \frac{1}{2 T_{k+1}^{2}} & -\frac{3}{T_{k+1}^{4}} & \frac{1}{T_{k+1}^{3}}
\end{array}\right] \cdot\left[\begin{array}{c}
P_{k} \\
v_{k} \\
a_{k} \\
P_{k+1} \\
v_{k+1}
\end{array}\right]
$$

The matrix equation above allows to evaluate the coefficients of the polynomial functions for the whole trajectory, with the exception of the last segment, which must take into account also the constraint on the continuity of acceleration:

$$
\begin{equation*}
a_{N-1}\left(T_{N}\right)=a_{N} \tag{5}
\end{equation*}
$$

Such segment is defined as the $5^{\text {th }}$ order polynomial function:

$$
\begin{gather*}
F_{N-1}(t)=B_{N-1,1}+B_{N-1,2} t+B_{N-1,3} t^{2}+ \\
+B_{N-1,4} t^{3}+B_{N-1,5} t^{4}+B_{N-1,6} t^{5} \tag{6}
\end{gather*}
$$

Therefore the boundary conditions for the last segment of the trajectory $\left(0 \leq t \leq T_{N-1}\right)$ are:

$$
\begin{align*}
& F_{N-1}(0)=B_{N-1,1}=P_{N-1} \\
& \quad v_{N-1}(0)=B_{N-1,2}=v_{N-1} \\
& a_{N-1}(0)=2 B_{N-1,3}=a_{N-1} \\
& F_{N-1}\left(T_{N}\right)=B_{N-1,1}+B_{N-1,2} T_{N}+B_{N-1,3} T_{N}^{2}+ \\
& \quad+B_{N-1,4} T_{N}^{3}+B_{N-1,5} T_{N}^{4}+B_{N-1,6} T_{N}^{5}=P_{N}  \tag{7}\\
& \quad v_{N-1}\left(T_{N}\right)=B_{N-1,2}+2 B_{N-1,3} T_{N}+3 B_{N-1,4} T_{N}^{2}+ \\
& \quad+4 B_{N-1,5} T_{N}^{3}+5 B_{N-1,6} T_{N}^{4}=v_{N} \\
& a_{N-1}\left(T_{N}\right)=2 B_{N-1,3}+6 B_{N-1,4} T_{N}+12 B_{N-1,5} T_{N}^{2}+ \\
& \quad+20 B_{N-1,6} T_{N}^{3}=a_{N}
\end{align*}
$$

All the constraints above can be described by a single matrix equation:

$$
\left[\begin{array}{l}
B_{N-1,1}  \tag{8}\\
B_{N-1,2} \\
B_{N-1,3} \\
B_{N-1,4} \\
B_{N-1,5} \\
B_{N-1,6}
\end{array}\right]=\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{10}{T_{N}^{3}} & -\frac{6}{T_{N}^{2}} & -\frac{3}{2 T_{N}} & \frac{10}{T_{N}^{3}} & -\frac{4}{T_{N}^{2}} & \frac{1}{2 T_{N}} \\
\frac{15}{T_{N}^{4}} & \frac{8}{T_{N}^{3}} & \frac{3}{2 T_{N}^{2}} & -\frac{15}{T_{N}^{4}} & \frac{7}{T_{N}^{3}} & -\frac{1}{T_{N}^{2}} \\
-\frac{6}{T_{N}^{5}} & -\frac{3}{T_{N}^{4}} & -\frac{1}{2 T_{N}^{3}} & \frac{6}{T_{N}^{5}} & -\frac{3}{T_{N}^{4}} & \frac{1}{2 T_{N}^{3}}
\end{array}\right] \cdot\left[\begin{array}{c}
P_{N-1} \\
v_{N-1} \\
a_{N-1} \\
P_{N} \\
v_{N} \\
a_{N}
\end{array}\right]
$$

The velocities $v_{k}$ and accelerations $a_{k}$ can be determined using the conditions of continuity of acceleration and jerk at the via-points. Given the three via-points $P_{k}, P_{k+1}$ and $P_{k+2}$ ( $1 \leq k \leq N-4$ ), the acceleration at the end of the first segment is:

$$
\begin{equation*}
a_{k}\left(T_{k+1}\right)=2 B_{k, 3}+6 B_{k, 4} T_{k+1}+12 B_{k, 5} T_{k+1}^{2} \tag{9}
\end{equation*}
$$

The value of jerk at the end of the first segment is:

$$
\begin{equation*}
j_{k}\left(T_{k+1}\right)=6 B_{k, 4}+24 B_{k, 5} T_{k+1} \tag{10}
\end{equation*}
$$

Similarly, the acceleration and the jerk at the beginning of the second segment are:

$$
\begin{equation*}
a_{k+1}(0)=2 B_{k+1,3} \tag{11}
\end{equation*}
$$

$$
\begin{equation*}
j_{k+1}(0)=6 B_{k+1,4} \tag{12}
\end{equation*}
$$

The constraint on continuity of acceleration requires that $a_{k}\left(T_{k+1}\right)=a_{k+1}(0)$, thus obtaining:

$$
\begin{equation*}
\frac{6}{T_{k+1}} v_{k}+a_{k}+\frac{6}{T_{k+1}} v_{k+1}-a_{k+1}=\frac{12}{T_{k+1}^{2}}\left(P_{k+1}-P_{k}\right) \tag{13}
\end{equation*}
$$

At the same way, continuity of jerk can be obtained using Eqn. 10 and 12 , leading to:

$$
\begin{align*}
\frac{30}{T_{k+1}^{2}} v_{k} & +\frac{6}{T_{k+1}} a_{k}+\left(\frac{18}{T_{k+1}^{2}}+\frac{18}{T_{k+2}^{2}}\right) v_{k+1}+\frac{6}{T_{k+2}} a_{k+2}+ \\
& +\frac{6}{T_{k+2}^{2}} v_{k+2}=\frac{48}{T_{k+1}^{3}}\left(P_{k+1}-P_{k}\right)+\frac{24}{T_{k+2}^{3}}\left(P_{k+2}-P_{k+1}\right) \tag{14}
\end{align*}
$$

$$
\begin{equation*}
a_{N-2}\left(T_{N-1}\right)=a_{N-1}(0) \tag{15}
\end{equation*}
$$

$$
\begin{equation*}
j_{N-2}\left(T_{N-1}\right)=j_{N-1}(0) \tag{16}
\end{equation*}
$$

Substituting the right values in Eqn. 15 and 16 leads to:

$$
\begin{equation*}
\frac{3}{T_{k+1}} v_{k}+\frac{1}{2} a_{k}+\frac{3}{T_{k+1}} v_{k+1}-\frac{1}{2} a_{k+1}=\frac{6}{T_{k+1}^{2}}\left(P_{k+1}-P_{k}\right) \tag{17}
\end{equation*}
$$

$$
\begin{align*}
& \frac{5}{T_{N-1}^{2}} v_{N-2}+\frac{1}{T_{N-1}} a_{N-2}+\left(\frac{3}{T_{N-1}^{2}}+\frac{6}{T_{N}^{2}}\right) v_{N-1}+\frac{3}{2 T_{N}} a_{N-1}= \\
& \quad=\frac{8}{T_{N-1}^{3}}\left(P_{N-1}-P_{N-2}\right)+\frac{10}{T_{N}^{3}}\left(P_{N}-P_{N-1}\right)-\frac{4}{T_{N}^{2}} v_{N}+\frac{1}{2 T_{N}} a_{N} \tag{18}
\end{align*}
$$

The results in Eqn. 13-18 can be merged in a single matrix equation. A matrix $\mathbf{M}$ can be defined and used for evaluating the $N-2$ unknown velocities and $N-2$ accelerations at the viapoints, i.e. the matrix $\mathbf{D} . \mathbf{H}$ is the vector of time intervals between two consecutive via points. Matrix $\mathbf{M}, \mathbf{D}$ and $\mathbf{H}$ have sizes, $(2 N-$ $4) \times(2 N-4),(2 N-4) \times 1$ and $(2 N-4) \times 1$, respectively.

$$
\begin{equation*}
\mathbf{M D}=\mathbf{H} \tag{19}
\end{equation*}
$$

$$
\mathbf{D}=\left[\begin{array}{c}
v_{2}  \tag{20}\\
a_{2} \\
v_{3} \\
a_{3} \\
\vdots \\
\\
v_{N-2} \\
a_{N-2} \\
v_{N-1} \\
a_{N-1}
\end{array}\right] \quad \mathbf{H}=\left[\begin{array}{c}
h_{1} \\
h_{2} \\
h_{3} \\
\vdots \\
\\
\\
\\
h_{2 N-5} \\
h_{2 N-4}
\end{array}\right]
$$

in which:

$$
\begin{align*}
& h_{1}=\frac{12}{T_{2}^{2}}\left(P_{2}-P_{1}\right)-\frac{6}{T_{2}} v_{1}-a_{1} \\
& h_{2}=\frac{48}{T_{2}^{3}}\left(P_{2}-P_{1}\right)+\frac{24}{T_{3}^{3}}\left(P_{3}-P_{2}\right)-\frac{30}{T_{2}^{2}} v_{1}-\frac{6}{T_{2}} a_{1} \\
& h_{2 k-1}=12\left(P_{k+1}-P_{k}\right), \quad k=2, \ldots, N-3 \\
& h_{2 k}=8 T_{K+2}^{3}\left(P_{k+1}-P_{k}\right)+4 T_{K+1}^{3}\left(P_{k+2}-P_{k+1}\right), \quad k=2, \ldots, N-3 \\
& h_{2 N-5}=\frac{6}{T_{N-1}^{2}}\left(P_{N-1}-P_{N-2}\right) \\
& h_{2 N-4}=\frac{8}{T_{N-1}^{3}}\left(P_{N-1}-P_{N-2}\right)+\frac{10}{T_{N}^{3}}\left(P_{N}-P_{N-1}\right)-\frac{4}{T_{N}^{2}} v_{N}+\frac{1}{2 T_{N}} a_{N} \tag{21}
\end{align*}
$$

The expression of matrix $\mathbf{M}$ is not reported here, due to the space constraints of the paper, but can be found in [10].

## The novel " 545 " trajectory

The novel " 545 " trajectory introduced in this paper has been developed in order to solve the problem of the nonzero jerk at the beginning of the trajectory. The ability to choose the initial jerk allows us to obtain a trajectory which is smooth during the critical phase of starting the motion of the robot from a rest condition. In real application, this translates in a lower level of vibration, a more accurate reproduction of the planned trajectory
and a lower mechanical stress to the robot structure during the whole task, comparing to the results that can be achieved with the " 445 " algorithm. On the other hand, numerical evidences shows that for the same kinematic constraints, i.e. limits on velocity, acceleration and jerk at the joints, a higher total execution time is obtained. will be shown in detail in Section $\mathbf{X}$.

In order to compute numerically a trajectory according to the novel " 545 " algorithm and using the same notation of Eqn.19, new expressions must be computed for matrices $\mathbf{H}, \mathbf{D}$ and $\mathbf{H}$. The changes form the " 445 " algorithm are:

1. the interpolating function for the first segment of the trajectory is now a $5^{\text {th }}$ order polynomial function: the new coefficients are shown in Eqn. 22.
2. the first two rows of matrix $\mathbf{M}$ and of the vector $\mathbf{H}$ are different: the new expressions are sown in Eqn. 23 and in Appendix A.

The new continuity conditions between the first and the second segment of the trajectory can be expressed according to the following matrix equation:

$$
\left[\begin{array}{l}
B_{1,1}  \tag{22}\\
B_{1,2} \\
B_{1,3} \\
B_{1,4} \\
B_{1,5} \\
B_{1,6}
\end{array}\right]=\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\
-\frac{5}{T_{2}^{4}} & -\frac{4}{T_{2}^{3}} & -\frac{3}{2 T_{2}^{2}} & -\frac{1}{3 T_{2}} & \frac{5}{T_{2}^{4}} & -\frac{1}{T_{2}^{3}} \\
\frac{4}{T_{2}^{5}} & \frac{3}{T_{2}^{4}} & \frac{1}{T_{2}^{3}} & \frac{1}{6 T_{2}^{2}} & -\frac{4}{T_{2}^{5}} & \frac{1}{T_{2}^{4}}
\end{array}\right] \cdot\left[\begin{array}{l}
P_{1} \\
v_{1} \\
a_{1} \\
j_{1} \\
P_{2} \\
v_{2}
\end{array}\right]
$$

The other continuity conditions are the same as the ones previously shown for the " 455 " method. Using the notation introduced in Eqn.19, the new values for the first two rows of array $\mathbf{H}$ are:

$$
\begin{align*}
& h_{1}=\frac{20}{T_{2}^{2}}\left(P_{1}-P_{2}\right)+\frac{12}{T_{2}} v_{1}+3 a_{1}+\frac{1}{3} T_{2} j_{1} \\
& h_{2}=\frac{120}{T_{2}^{3}}\left(P_{1}-P_{2}\right)+\frac{24}{T_{3}^{3}}\left(P_{2}-P_{3}\right)+\frac{84}{T_{2}^{2}} v_{1}+\frac{24}{T_{2}} a_{1}+3 j_{1} \tag{23}
\end{align*}
$$

The correct formulation for matrix $\mathbf{D}$ is reported in Appendix A .

## The " 5455 " trajectory

The novel " 5455 " trajectory is meant to be an improved version of trajectory " 545 ". This primitive function is based on the use of $5^{\text {th }}$ order polynomial functions for the first and the last two
segments of the trajectory. All the other segments are described by $4^{\text {th }}$ order polynomial functions. In this way, it is possible to achieve the following goals:

1. jerk is continuous for the whole trajectory
2. the value of jerk at the first and last via point can be imposed

It should be pointed out that with the " 445 " trajectory the jerk at the first and last via points cannot be imposed arbitrarily. On the other hand, the " 545 " trajectory only allows to choose the value of the jerk only at the first via point. Therefore the choice of a " 5455 " trajectory sports an extra degree of freedom, in comparison to the " 545 ". When planning a " 5455 " trajectory, the user might choose to achieve a null jerk at the extremities of the path (i.e. $j_{0}=0$ and $j_{N}=0$ ), or he can choose to impose two arbitrary values for $j_{0}$ and $j_{N}$. The first choice has more appeal when maximum smoothness is the goal, while the second one can be efficiently exploited when the trajectory planning includes a composite trajectory. One example of a profitable use of nonzero initial and final jerk is the planning of trajectories with repetitive tasks. In this case, the user might want to plan the whole trajectory in this way:

1. plan the trajectory for the first cycle with $j_{0}=0$ and $j_{N}=j^{*}$
2. plan all the intermediate cycles with $j_{0}=j_{N}=j^{*}$
3. plan the last cycle with $j_{0}=j^{*}$ and $j_{N}=0$

In this way a complex trajectory with repetitive cycles can be planned obtaining zero initial and final jerk, and continuity of jerk can be achieved along the whole path as well. This goal cannot be achieved neither with the " 445 ", nor with the " 5455 " trajectory.

The continuity conditions for the " 5455 " trajectory are the same as the ones for the " 545 ", except for the last two segments of the trajectory. The new continuity conditions expressed in matrix form are:

$$
\left[\begin{array}{c}
B_{N-2,1}  \tag{24}\\
B_{N-2,2} \\
B_{N-2,3} \\
B_{N-2,4} \\
B_{N-2,5} \\
B_{N-2,6}
\end{array}\right]=\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{10}{T_{N-1}^{3}} & -\frac{6}{T_{N-1}^{2}} & -\frac{3}{2 T_{N-1}} & \frac{10}{T_{N-1}^{3}} & -\frac{4}{T_{N-1}^{2}} & \frac{1}{2 T_{N-1}} \\
\frac{15}{T_{N-1}^{4}} & \frac{8}{T_{N-1}^{3}} & \frac{3}{2 T_{N-1}^{2}} & -\frac{15}{T_{N-1}^{4}} & \frac{7}{T_{N-1}^{3}} & -\frac{1}{T_{N-1}^{2}} \\
-\frac{6}{T_{N-1}^{S}} & -\frac{1}{T_{N-1}^{4}} & -\frac{1}{2 T_{N-1}^{3}} & \frac{6}{T_{N-1}^{3}} & -\frac{3}{T_{N-1}^{4}} & \frac{1}{2 T_{N-1}^{3}}
\end{array}\right] \cdot\left[\begin{array}{c}
P_{N-2} \\
v_{N-2} \\
a_{N-2} \\
P_{N-1} \\
v_{N-1} \\
a_{N-1}
\end{array}\right]
$$

$$
\left[\begin{array}{l}
B_{N-1,1}  \tag{25}\\
B_{N-1,2} \\
B_{N-1,3} \\
B_{N-1,4} \\
B_{N-1,5} \\
B_{N-1,6}
\end{array}\right]=\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{10}{T_{N}^{2}} & -\frac{4}{T_{N}} & \frac{10}{T_{N}^{2}} & -\frac{6}{T_{N}} & \frac{3}{2} & -\frac{1}{6 T_{N}} \\
\frac{20}{T_{N}^{3}} & \frac{6}{T_{N}^{2}} & -\frac{20}{T_{N}^{3}} & \frac{14}{T_{N}^{2}} & -\frac{4}{T_{N}} & \frac{1}{2} \\
-\frac{15}{T_{N}^{4}} & -\frac{4}{T_{N}^{3}} & \frac{15}{T_{N}^{4}} & -\frac{11}{T_{N}^{3}} & \frac{7}{2 T_{N}^{2}} & -\frac{1}{2 T_{N}} \\
\frac{4}{T_{N}^{5}} & \frac{1}{T_{N}^{4}} & -\frac{4}{T_{N}^{5}} & \frac{3}{T_{N}^{4}} & -\frac{1}{T_{N}^{3}} & \frac{1}{6 T_{N}^{2}}
\end{array}\right] \cdot\left[\begin{array}{c}
P_{N-1} \\
v_{N-1} \\
P_{N} \\
v_{N} \\
a_{N} \\
j_{N}
\end{array}\right]
$$

$$
\begin{align*}
h_{2 N-7}= & \frac{6}{T_{N-2}^{2}}\left(P_{N-2}-P_{N-3}\right) \\
h_{2 N-6}= & \frac{8}{T_{N-2}^{3}}\left(P_{N-2}-P_{N-3}\right)+\frac{10}{T_{N-1}^{3}}\left(P_{N-1}-P_{N-2}\right) \\
h_{2 N-5}= & \frac{20}{T_{N}^{2}}\left(P_{N}-P_{N-1}\right)-\frac{12}{T_{N}} v_{N}+3 a_{N}-\frac{1}{3} T_{N} j_{N}  \tag{26}\\
h_{2 N-4}= & \frac{20}{T_{N-1}^{3}} P_{N-2}-\left(\frac{20}{T_{N-1}^{3}}-\frac{40}{T_{N}^{3}}\right) P_{N-1}-\frac{40}{T_{N}^{3}} P_{N}+ \\
& +\frac{84}{3 T_{N}^{2}} v_{N}-\frac{8}{T_{N}} a_{N}+j_{N}
\end{align*}
$$

The above conditions can be used to compute the new correct formulation for matrix $\mathbf{M}$, which is reported in Appendix A.

## NUMERICAL RESULTS

The objective of the trajectory generation problem considered in this paper is to produce a trajectory in the joint space, given a set of via-points defined in the operative space. The resulting trajectory must respect some kinematic constraints, and be optimal in the sense of a given cost function. The kinematic constraints are used to adapt the algorithm to the specific robotic manipulator under consideration. This is a feature of paramount importance in all industrial applications, where the datasheet values of the manipulator must be carefully taken into account for many reasons, including safety. Here it has been chosen to set hard constraints on the maximum and minimum values for the velocity, the acceleration and for the jerk at each joint. It should be pointed out that a limit on the maximum value of the jerk can also be used to control the smoothness of the trajectory. Since the jerk limit can be chosen freely, trajectories with arbitrarily high smoothness can be achieved, at the cost of obtaining an higher execution time. For the tests presented in the following section, symmetrical limits have been used. Moreover identical values have been used for all the joints, but the algorithm developed by the Authors allows to use also limit values which are not symmetrical and not uniform for all the joints.

The cost function of choice is the well known minimum time, i.e. the objective of the trajectory planning algorithm is to generate the trajectory which has the minimum overall execution time, within the constraints mentioned above. Other choices are possible too, such as minimum energy, minimum jerk, or minimum time-jerk, as in [13].

The optimization problem can be stated as:

$$
\begin{cases}\min _{\mathbf{h}} \sum_{i=1}^{v_{p}-1} h_{i} &  \tag{27}\\ \text { subject to: } & \\ & \\ V_{\min } \leq \dot{q}_{j}(t) \leq V_{\max } & j=1 \ldots N \\ A_{\min } \leq \ddot{q}_{j}(t) \leq A_{\max } & j=1 \ldots N \\ J_{\min } \leq \ddot{q}_{j}(t) \leq J_{\max } & j=1 \ldots N\end{cases}
$$

with:

$$
\begin{equation*}
\mathbf{h}=\left[h_{1}, h_{2}, \ldots, h_{v_{p}-1}\right] \tag{28}
\end{equation*}
$$

TABLE 1. NOMENCLATURE FOR EQN. 27

| Symbol | Definition |
| :---: | :--- |
| $N$ | Number of robot joints |
| $v_{p}$ | Number of via-points |
| $h_{i}$ | Time interval between two via-points |
| $\dot{q}_{j}(t)$ | Velocity of the $j$ th joint |
| $\ddot{q}_{j}(t)$ | Acceleration of the $j$ th joint |
| $\dddot{q}_{j}(t)$ | Jerk of the $j$ th joint |
| $V_{\min }, V_{\max }$ | Velocity limits for the $j$ th joint |
| $A_{\min }, A_{\max }$ | Acceleration limits for the $j$ th joint |
| $J_{\min }, J_{\max }$ | Jerk limits for the $j$ th joint |

Once a set of via points in the operative space is defined, the trajectory planning algorithm proposed in this paper follows the sequence of steps:

1. the set of via-points in the joint space is automatically generated, by means of a kinematic inversion routine


FIGURE 1. PROTOTYPE OF THE CARTESIAN ROBOT, MECHATRONICS LAB, UNIVERSITY OF UDINE, ITALY
2. the speed, acceleration and jerk constraints according to the manipulator structural constraints and to the required level of smoothness are set by the used
3. an initial solution for the vector of time intervals $\mathbf{h}$ is provided to the algorithm
4. the optimization problem defined in Eqn. 27 is solved by a specific program. The outcast of the problem is the vector $h$ of optimal time intervals

All the numerical results presented are based on a real prototype of the 3 axes Cartesian robot available in the Mechatronics Lab at University of Udine, Italy, depicted in Figure 1. The exact values of the kinematics constraints are reported in Table 2. Such robot is controlled by commercial PLC modules, as most industrial robots, and is operated by feeding the control with a sequence of position values for each individual joint.

In this section a comparison between the results obtainable with several planning algorithms are reported, in order to provide a comparison of the performance of the novel approaches proposed in this paper. The first test is run choosing a pick \& place task specified by 8 via-points. Figures $2-5$ show the evolution of position, velocity and acceleration for the Y axis of the robot. The results for the X and Z axis are not shown, due to the limited space of the paper. A comparison is established compar-

TABLE 2. KINEMATIC CONSTRAINTS

|  | Min value | Max value |
| :---: | :--- | :--- |
| Velocity | $-225 \mathrm{~mm} / \mathrm{s}$ | $225 \mathrm{~mm} / \mathrm{s}$ |
| Acceleration | $-2400 \mathrm{~mm} / \mathrm{s}^{2}$ | $2400 \mathrm{~mm} / \mathrm{s}^{2}$ |
| Jerk | $-2400 \mathrm{~mm} / \mathrm{s}^{3}$ | $2400 \mathrm{~mm} / \mathrm{s}^{3}$ |

ing the " 545 " and " 5455 " with the " 434 " and " 445 " algorithms with minimum time, with bounded speed, acceleration and jerk. As it can be seen in Figure 5, continuous jerk cannot be obtained by the " 434 ", while the other methods achieve this goal. Null jerk value can be obtained at the initial point by the " 545 " and the "5455", and the latter shows also a zero final jerk value. The price to be paid for the superior level of smoothness is clearly a longer total execution time. The beneficial effects on smoothness provided by the " 5455 " trajectory is clearly shown in figure 5: the " 5455 " has the lowest overall peak jerk, which is equal to $1932 \mathrm{~mm} / \mathrm{s}^{3}$, while the peak value of jerk for the " 445 " and " 545 " methods is limited to $2500 \mathrm{~mm} / \mathrm{s}^{3}$, as such value is limited by the jerk constraints imposed by the user.

In order to provide a more quantitative comparison between the proposed methods and some other popular methods available in literature, the results of other several tests are reported in Figure 6 and 7. The set of via points taken into consideration for each individual tests are are disposed along the vertexes of a triangle, of a square and along a circle in the operative space. Results are provided in terms of total execution time and integral of the quadratic norm of the jerk, which is a good indicator of the smoothness of the whole trajectory, since it takes into account the jerk values along the whole trajectory, not just the peak values. The BSPL5J and SPL3J methods, developed in [8], are included as well.

The comparison shows that within similar execution time, the " 5455 " trajectory shows an overall integral of jerk value which is outperformed only by the BSPL5J method. Considering the case of the square task, the " 5455 " can achieves the lowest overall value of the integral of jerk, but with the longest execution time. The " 545 " trajectory shows a performance level which appears to be a good compromise between the speed of execution of the task of the " 445 " and the smoothness of the " 5455 ".

## CONCLUSION

In this paper two new trajectory planning algorithms have been presented. The first one, named " 545 " allows to interpolate a set of $N$ via-points obtaining the continuity of jerk along the whole path. Moreover, the value of jerk at the initial point can be arbitrarily imposed. The other novel trajectory introduced in this


FIGURE 2. COMPARISON BETWEEN 434, 445, 454 AND 5455 ALGORITMS FOR A PICK \& PLACE TRAJECTORY TRAJECTORIES: POSITION OF Y AXIS


FIGURE 3. COMPARISON BETWEEN 434, 445, 454 AND 5455 ALGORITMS FOR A PICK \& PLACE TRAJECTORY TRAJECTORIES: SPEED OF Y AXIS
paper is the " 5455 ", which ensures the continuity of jerk while allowing the user to decide the value of jerk at both the first and last via-point. This feature can be efficiently used to plan trajectories with repeated cycles retaining the continuity of jerk along the whole task, with initial and final null jerk. Such characteristics cannot be obtained neither with the " 445 ", nor with the " 545 " method. Comparison with other trajectory planning algorithms prove that the proposed approaches can achieve a good compromise between the level of smoothness and the total exe-


FIGURE 4. COMPARISON BETWEEN 434, 445, 454 AND 5455 ALGORITMS FOR A PICK \& PLACE TRAJECTORY TRAJECTORIES: ACCELERATION ALONG Y AXIS


FIGURE 5. COMPARISON BETWEEN 434, 445, 454 AND 5455 ALGORITMS FOR A PICK \& PLACE TRAJECTORY TRAJECTORIES: JERK ALONG Y AXIS
cution time, when minimal time trajectories with bounded speed, acceleration and jerk are taken into account.

## ACKNOWLEDGMENT

Sincere thanks go to A. Lanzutti, M. Martin and V. Zanotto for their contribution to this work.


FIGURE 6. TOTAL EXECUTION TIME FOR SEVERAL TRAJECTORIES


FIGURE 7. INTEGRAL OF JERK FOR SEVERAL TRAJECTORIES

## REFERENCES

[1] Biagiotti, L., and Melchiorri, C., 2008. Trajectory planning for automatic machines and robots. Springer Verlag.
[2] Valero, F., Mata, V., and Besa, A., 2006. "Trajectory planning in workspaces with obstacles taking into account the dynamic robot behaviour". Mechanism and machine theory, 41(5), pp. 525-536.
[3] Piazzi, A., and Visioli, A., 2000. "Global minimum-jerk trajectory planning of robot manipulators". Industrial Electronics, IEEE Transactions on, 47(1), feb, pp. 140-149.
[4] Barre, P., Bearee, R., Borne, P., and Dumetz, E., 2005. "Influence of a jerk controlled movement law on the vibratory
behaviour of high-dynamics systems". Journal of Intelligent \& Robotic Systems, 42(3), pp. 275-293.
[5] Dumetz, E., Dieulot, J., Barre, P., Colas, F., and Delplace, T., 2006. "Control of an industrial robot using acceleration feedback". Journal of Intelligent \& Robotic Systems, 46(2), pp. 111-128.
[6] Zefran, M., Kumar, V., and Croke, C., 1998. "On the generation of smooth three-dimensional rigid body motions". Robotics and Automation, IEEE Transactions on, 14(4), pp. 576-589.
[7] Gasparetto, A., and Zanotto, V., 2007. "A new method for smooth trajectory planning of robot manipulators". Mechanism and Machine Theory, 42(4), pp. 455-471.
[8] Gasparetto, A., and Zanotto, V., 2008. "A technique for time-jerk optimal planning of robot trajectories". Robotics and Computer-Integrated Manufacturing, 24(3), pp. 415426.
[9] Ho, C., and Cook, C., 1982. "The application of spline functions to trajectory generation for computer controlled manipulators". Digital Systems for Industrial Automation, 1(4), pp. 325-333.
[10] Petrinec, K., and Kovacic, Z., 2007. "Trajectory planning algorithm based on the continuity of jerk". In Control \& Automation, 2007. MED'07. Mediterranean Conference on, IEEE, pp. 1-5.
[11] Boryga, M., and Grabos, A., 2009. "Planning of manipulator motion trajectory with higher-degree polynomials use". Mechanism and machine theory, 44(7), pp. 1400-1419.
[12] Lin, C., Chang, P., and Luh, J., 1983. "Formulation and optimization of cubic polynomial joint trajectories for industrial robots". Automatic Control, IEEE Transactions on, 28(12), pp. 1066-1074.
[13] Zanotto, V., Gasparetto, A., Lanzutti, A., Boscariol, P., and Vidoni, R., 2011. "Experimental validation of minimum time-jerk algorithms for industrial robots". Journal of Intelligent \& Robotic Systems, pp. 1-23.
[14] Tangpattanakul, P., and Artrit, P., 2009. "Minimum-time trajectory of robot manipulator using harmony search algorithm". In Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology, 2009. ECTI-CON 2009. 6th International Conference on, Vol. 1, IEEE, pp. 354-357.
[15] Wang, C., and Horng, J., 1990. "Constrained minimumtime path planning for robot manipulators via virtual knots of the cubic b-spline functions". Automatic Control, IEEE Transactions on, 35(5), pp. 573-577.
[16] Sahar, G., and Hollerbach, J., 1986. "Planning of minimum-time trajectories for robot arms". The International journal of robotics research, 5(3), pp. 90-100.

## Appendix A: Matrix M for the " 545 " and " 5455 " trajectories





[^0]:    *Address all correspondence to this author.

