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DRAFT: JERK-CONTINOUS TRAJECTORIES FOR CYCLIC TASKS

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ABSTRACT

Planning smooth trajectories is a crucial task for most advanced robotic applications. Poorly planned trajectories can be inefficient under many aspects, since they might require long execution time and induce unnecessary vibration on the end-effector of the robot as well as high solicitation on its mechanical structure and actuators. In this paper a novel trajectory planning methods for robotic manipulators is introduced, named "5455". This method is based on an interpolation of a sequence of via points using a combination of 4^{th} *and* 5^{th} *order polynomial functions. This technique allows to obtain a continuous-jerk trajectory for improved smoothness and minimum excitation of vibration. Such method allows also to impose an arbitrary value of jerk at the first and last via-point. This feature can be effectively used to produce a smooth trajectory for repetitive tasks, trough an innovative optimization algorithm which is introduced in this paper. Both numerical and experimental results are provided for a 3 d.o.f. Cartesian robot, but the techniques provided here can be applied to any industrial manipulator.*

INTRODUCTION

The ever growing market of industrial robot applications requires the development of high performance trajectory planning algorithms. Severe vibrations arise in manipulators when they are moved along a non-smooth trajectory. In that case premature joint wear and mechanical failures might occur, as well as a major worsening of the motion accuracy. Therefore in the last decades a large number of techniques have been developed to design smooth trajectories for industrial robots. An extended view of the problem can be found in the book [1]. Both online and offline techniques have been developed: this work falls into the latter category. Another fundamental distinction between the several methods available in literature is the use of a model-based or of a model-free approach. While model-based approaches can achieve good results in a specific case [2], they lack the generality which is a fundamental requirement for most industrial application. As a matter of fact most industrial facilities does not have the knowledge required to work with model-based approaches, and the investment on personnel training is not reputed to be profitable. Therefore model-free approaches, as the one presented in this paper, are much more appealing for today's market.

In this paper a novel trajectory planning algorithm for industrial robot is presented. This algorithm produces an optimal trajectory starting from the definition of a set of via-points, i.e. it adjusts the time distance between two consecutive via points in order to minimize a cost function of choice. Constraints on velocity, acceleration and jerk at each joint can be specified as inputs of the optimization procedure. Similar approaches can be encountered quite often in literature, since many methods are

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available to produce a time law which interpolates or approximates a set of via-points. These works can be classified on the base of different features, such as the choice of the cost function and of the primitive functions used for the interpolation procedure. The paper [3] introduces a method for the evaluation of minimum jerk trajectories as a global constrained minimax optimization problem. The interpolation is based on a sequence of cubic polynomial functions. Continuous jerk can be achieved by this method, but this procedure has very high computational demands. Fifth order B-spline together with a composite time-jerk cost function are used in [4, 5]. Other popular techniques are based on simpler interpolation functions, such as the ”343” approach developed by Ho and Cook in [6]. This method relies on the use of spline functions, but does not guarantee the continuity of jerk along the trajectory. The problem of continuity of jerk has been solved by Petrinec and Kovacic in [7], trough the development of the ”445” trajectory. Such approach makes use of an interpolation function composed by a sequence of 4^{th} order polynomial functions, with a 5^{th} order function for interpolating the motion between the last two via-points. The use of polynomial functions of even high order allows to produce motion profiles with superior smoothness, but at the cost of a general increase of peak values of peak speed, if a constant total execution time is considered. An example of the use of polynomials functions up to the 9^{th} order can be found in [8].

Several papers are focused on keeping the value of jerk limited and possibly continuous, since jerk heavily affects the accuracy of motion and the solicitation to the mechanical structure of the robot. Moreover, smooth trajectories are to be preferred in order to comply with the bandwidth limitation of the actuators. The correlation between trajectory smoothness and the performance of a robotic system has been demonstrated both theoretically [9] and experimentally [10] in several papers.

The novel method presented here is based on a new composite trajectory primitive, called ”5455”. It uses a 5^{th} order polynomial function as the primitive for the first and the last two segments of the trajectory, while the intermediate segments are described by 4^{th} order polynomial functions. This choice, together with a suitable choice of the continuity conditions at each via-point, ensures that:

1. jerk is continuous along the whole trajectory
2. an arbitrary jerk value can be specified for the first and the last via-point

While the first feature is shared with other algorithms, such as the ones that are based on cubic splines [11, 12], many approaches do not allow for the operator to choose all the kinematic parameters for the first and the last point of the trajectories. As it will be shown in the paper, this feature allows to produce trajectories with null acceleration and jerk at the extreme point of the trajectory, a feature that is reputed to be very important in terms of vibration reduction, as highlighted by Boryga et al. in [8]. This feature can be efficiently used to maximize the smoothness of the motion profile during the critical phase of start-up and resting of the robot. Moreover the free choice of initial and final jerk allows to retain maximum smoothness even in the case of the cyclical repetition of the same task.

The evaluation of the effectiveness of the proposed trajectory is conducted by means of extensive experimental tests. A three degrees-of-freedom Cartesian manipulator is chosen for testing the novel approach proposed here. Numerical evidences confirm that the proposed approach achieves a good level of performance when compared to other popular trajectory planning algorithms [3, 4, 7, 13]. The cost function on which the optimization procedure is based is the well known minimum time, which has proved to be a popular approach [14–16], given the appeal of its potential economic advantage for most industrial applications. Here only the case of minimum time with kinematic constraints is considered, but the ”5455” primitive can be used as a starting point for other innovative methods, simply by changing the choice of the goal function. The optimization problem can be solved easily and with limited computational effort with several optimization routines, such as Matlab’s *fmincon*. Constraints can be handled with ease, since the kinematics limits can be expressed as an explicit function of the time intervals between two consecutive via points.

1 THE ”5455” TRAJECTORY

In this section the novel ”5455” trajectory planning algorithm will be introduced. Another new algorithm, named ”545” will be introduced as well. These two methods are based on a different sequence of polynomial functions. Both algorithms allow for the use of hard constraints on velocity, acceleration and jerk of each joint of the robot independently, thus allowing their use for virtually every industrial robot.

The ”545” trajectory

The ”545” trajectory planning method make use of a sequence of 4^{th} order polynomial function for interpolating the trajectory between the second and the next-to-last via point. The remaining first and last segment of the trajectory is interpolated using 5^{th} order polynomial function.

Let us indicate with P_k, v_k, a_k and j_k the position, velocity, acceleration and jerk at the k-th via-point, respectively. The first segment of the trajectory is, according to the proposed method, interpolated trough a polynomial of degree 5, as in:

$$F_k(t) = B_{k,1} + B_{k,2}t + B_{k,3}t^2 + B_{k,4}t^3 + B_{k,5}t^4 + B_{k,6}t^5 \qquad (1)$$

The symbol $F_k(t)$ represents the position of th k-th joint of the robot as a function of time. The boundary conditions imposed on the trajectory along the first segment are:

$$\begin{aligned}
F_k(0) &= P_1 \\
v_k(0) &= v_1 \\
a_k(0) &= a_1 \\
j_k(0) &= j_1 \\
F_k(T_2) &= P_2 \\
v_k(T_2) &= v_2
\end{aligned} \tag{2}$$

It should be noticed that the current choice of boundary conditions allows to choose the value of the jerk at the starting point of the trajectory, j_1. This feature is generally used to plan trajectories with initial null jerk. Substituting Eq. 2 into Eq. 1 allows to evaluate the $B_{1,k}$ coefficients as:

$$\begin{aligned}
&F_k(0) = B_{k,1} = P_1 \\
&v_k(0) = B_{k,2} = v_1 \\
&a_k(0) = 2B_{k,3} = a_1 \\
&j_k(0) = 6B_{k,4} = j_1 \\
&F_k(T_2) = B_{k,1} + B_{k,2}T_{k+1} + B_{k,3}T_{k+1}^2 + +B_{k,4}T_{k+1}^3 + B_{k,5}T_{k+1}^4 = P_2 \\
&v_k(T_2) = B_{k,2} + 2B_{k,3}T_{k+1} + 3B_{k,4}T_{k+1}^2 + 4B_{k,5}T_{k+1}^3 = v_2
\end{aligned} \tag{3}$$

The 6 equations in Eq. 12 can be merged in single matrix expression which shows the explicit expression of the polynomial coefficients $B_{1,1}, B_{1,2} \dots B_{1,6}$:

$$\begin{bmatrix} B_{1,1} \\ B_{1,2} \\ B_{1,3} \\ B_{1,4} \\ B_{1,5} \\ B_{1,6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\ -\frac{5}{T_2^4} & -\frac{4}{T_2^3} & -\frac{3}{2T_2^2} & -\frac{1}{3T_2} & \frac{5}{T_2^4} & -\frac{1}{T_2^3} \\ \frac{4}{T_2^5} & \frac{3}{T_2^4} & \frac{1}{T_2^3} & \frac{1}{6T_2^2} & -\frac{4}{T_2^5} & \frac{1}{T_2^4} \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ v_1 \\ a_1 \\ j_1 \\ P_2 \\ v_2 \end{bmatrix} \tag{4}$$

The intermediate segments of the trajectory are interpolated using the fourth order polynomial function:

$$F_k(t) = B_{k,1} + B_{k,2}t + B_{k,3}t^2 + B_{k,4}t^3 + B_{k,5}t^4 \tag{5}$$

which is valid for $2 \leq k \leq N-2$, being N the number of via-points. Since we are dealing with a four order polynomial, we need to impose 5 boundary conditions:

$$\begin{aligned}
F_k(0) &= P_k \\
v_k(0) &= v_k \\
a_k(0) &= a_k \\
F_k(T_{k+1}) &= P_{k+1} \\
v_k(T_{k+1}) &= v_{k+1}
\end{aligned} \tag{6}$$

Substituting the boundary conditions into Eq. 5 one obtains:

$$\begin{aligned}
&F_k(0) = B_{k,1} = P_k \\
&v_k(0) = B_{k,2} = v_k \\
&a_k(0) = 2B_{k,3} = a_k \\
&F_k(T_{k+1}) = B_{k,1} + B_{k,2}T_{k+1} + B_{k,3}T_{k+1}^2 + B_{k,4}T_{k+1}^3 + B_{k,5}T_{k+1}^4 = P_{k+1} \\
&v_k(T_{k+1}) = B_{k,2} + 2B_{k,3}T_{k+1} + 3B_{k,4}T_{k+1}^2 + 4B_{k,5}T_{k+1}^3 = v_{k+1}
\end{aligned} \tag{7}$$

The expression above can be written in matrix form, showing the explicit values of the B coefficients:

$$\begin{bmatrix} B_{k,1} \\ B_{k,2} \\ B_{k,3} \\ B_{k,4} \\ B_{k,5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ -\frac{4}{T_{k+1}^3} & -\frac{3}{T_{k+1}^2} & -\frac{1}{T_{k+1}} & \frac{4}{T_{k+1}^3} & -\frac{1}{T_{k+1}^2} \\ \frac{3}{T_{k+1}^4} & \frac{2}{T_{k+1}^3} & \frac{1}{2T_{k+1}^2} & -\frac{3}{T_{k+1}^4} & \frac{1}{T_{k+1}^3} \end{bmatrix} \cdot \begin{bmatrix} P_k \\ v_k \\ a_k \\ P_{k+1} \\ v_{k+1} \end{bmatrix} \tag{8}$$

The trajectory between the last and the next-to-last via-point is again represented by a polynomial of order 5. Therefore we need to add one condition to the one presented in Eq. 6. The resulting boundary conditions are:

$$\begin{aligned}
F_{N-1}(0) &= B_{N-1,1} = P_{N-1} \\
v_{N-1}(0) &= B_{N-1,2} = v_{N-1} \\
a_{N-1}(0) &= 2B_{N-1,3} = a_{N-1} \\
F_{N-1}(T_N) &= P_N \\
v_{N-1}(T_N) &= v_N \\
a_{N-1}(T_N) &= a_N
\end{aligned} \tag{9}$$

Notice that in this case we cannot impose an arbitrary value on the j_N, i.e. the jerk at the last via-point. Using the same procedure applied before, we use the polynomial form for expressing the positions, the velocities and the accelerations in Eq. 9:

$$
\begin{aligned}
&F_{N-1}(0) = B_{N-1,1} = P_{N-1} \\
&v_{N-1}(0) = B_{N-1,2} = v_{N-1} \\
&a_{N-1}(0) = 2B_{N-1,3} = a_{N-1} \\
&F_{N-1}(T_N) = B_{N-1,1} + B_{N-1,2}T_N + B_{N-1,3}T_N^2 + B_{N-1,4}T_N^3 \\
&\qquad + B_{N-1,5}T_N^4 + B_{N-1,6}T_N^5 = P_N \\
&v_{N-1}(T_N) = B_{N-1,2} + 2B_{N-1,3}T_N + 3B_{N-1,4}T_N^2 + 4B_{N-1,5}T_N^3 \\
&\qquad + 5B_{N-1,6}T_N^4 = v_N \\
&a_{N-1}(T_N) = 2B_{N-1,3} + 6B_{N-1,4}T_N + 12B_{N-1,5}T_N^2 \\
&\qquad + 20B_{N-1,6}T_N^3 = a_N
\end{aligned}
\tag{10}
$$

Again, the matrix form of the equation above is:

$$
\begin{bmatrix} B_{N-1,1} \\ B_{N-1,2} \\ B_{N-1,3} \\ B_{N-1,4} \\ B_{N-1,5} \\ B_{N-1,6} \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{10}{T_N^3} & -\frac{6}{T_N^2} & -\frac{3}{2T_N} & \frac{10}{T_N^3} & -\frac{4}{T_N^2} & \frac{1}{2T_N} \\
\frac{15}{T_N^4} & \frac{8}{T_N^3} & \frac{3}{2T_N^2} & -\frac{15}{T_N^4} & \frac{7}{T_N^3} & -\frac{1}{T_N^2} \\
-\frac{6}{T_N^5} & -\frac{3}{T_N^4} & -\frac{1}{2T_N^3} & \frac{6}{T_N^5} & -\frac{3}{T_N^4} & \frac{1}{2T_N^3}
\end{bmatrix}
\cdot
\begin{bmatrix} P_{N-1} \\ v_{N-1} \\ a_{N-1} \\ P_N \\ v_N \\ a_N \end{bmatrix}
\tag{11}
$$

The equations above, specifically Eq. 7,8,11, are not sufficient to evaluate all the kinematic parameters at every via-point. Acceleration and jerk at the second via-point can be evaluated trough the 2^{nd} and 3^{rd} time derivative of Eq. 1:

$$
\begin{aligned}
a_2 &= 2B_{1,3} + 6B_{1,4}T_2 + 12B_{1,5}T_2^2 + 20B_{1,6}T_2^3 \\
j_2 &= 6B_{1,4} + 24B_{1,5}T_2 + 60B_{1,6}T_2^2
\end{aligned}
\tag{12}
$$

The velocities and the accelerations at the other via-point can be evaluated using the continuity conditions of acceleration and jerk. If the three consecutive via-points P_k, P_{k+1} and P_{k+2} are considered, the acceleration at end of the first segment is:

$$
a_k(T_{k+1}) = 2B_{k,3} + 6B_{k,4}T_{k+1} + 12B_{k,5}T_{k+1}^2 \tag{13}
$$

while the jerk at the same point is:

$$
j_k(T_{k+1}) = 6B_{k,4} + 24B_{k,5}T_{k+1} \tag{14}
$$

Acceleration and jerk at the beginning of the second segment are:

$$
a_{k+1}(0) = 2B_{k+1,3} \tag{15}
$$

$$
j_{k+1}(0) = 6B_{k+1,4} \tag{16}
$$

Continuity of acceleration and jerk impose that Eq. 13 must be equal to Eq. 14, and that Eq. 15 must be equal to Eq. 16. Thus we obtain, respectively:

$$
\frac{6}{T_{k+1}}v_k + a_k + \frac{6}{T_{k+1}}v_{k+1} - a_{k+1} = \frac{12}{T_{k+1}^2}(P_{k+1} - P_k) \tag{17}
$$

$$
\begin{aligned}
&\frac{30}{T_{k+1}^2}v_k + \frac{6}{T_{k+1}}a_k + \left(\frac{18}{T_{k+1}^2} + \frac{18}{T_{k+2}^2}\right)v_{k+1} + \frac{6}{T_{k+2}}a_{k+2} + \frac{6}{T_{k+2}^2}v_{k+2} \\
&= \frac{48}{T_{k+1}^3}(P_{k+1} - P_k) + \frac{24}{T_{k+2}^3}(P_{k+2} - P_{k+1})
\end{aligned}
\tag{18}
$$

Continuity of acceleration and jerk for the last two segments can be expressed as:

$$
a_{N-2}(T_{N-1}) = a_{N-1}(0) \tag{19}
$$

$$
j_{N-2}(T_{N-1}) = j_{N-1}(0) \tag{20}
$$

Using the right values into Eq. 19 and 20 we obtain:

$$
\frac{3}{T_{k+1}}v_k + \frac{1}{2}a_k + \frac{3}{T_{k+1}}v_{k+1} - \frac{1}{2}a_{k+1} = \frac{6}{T_{k+1}^2}(P_{k+1} - P_k) \tag{21}
$$

$$\frac{5}{T_{N-1}^2}v_{N-2}+\frac{1}{T_{N-1}}a_{N-2}+(\frac{3}{T_{N-1}^2}+\frac{6}{T_N^2})v_{N-1}+\frac{3}{2T_N}a_{N-1}= \tag{22}$$

$$=\frac{8}{T_{N-1}^3}(P_{N-1}-P_{N-2})+\frac{10}{T_N^3}(P_N-P_{N-1})-\frac{4}{T_N^2}v_N+\frac{1}{2T_N}a_N$$

The results provided above can be used to determine a single matrix expression which allows to evaluate all the kinematic variables that are not imposed with the definition of the trajectory planning problem. We can define the matrix **M**, a vector of unknown velocities and accelerations **D**, whose length is $N-2$, and a vector of coefficients **H**. **M** is defined so that the following relationship holds:

$$\mathbf{M}\cdot\mathbf{D}=\mathbf{H} \tag{23}$$

Matrix **M** has size $(2N-4)\times(2N-4)$. The expressions of **D** and **H** are reported below. The expression of matrix **M** is reported in Appendix A.

$$D=\begin{bmatrix} v_2\\ a_2\\ v_3\\ a_3\\ \vdots\\ \\ v_{N-2}\\ a_{N-2}\\ v_{N-1}\\ a_{N-1}\end{bmatrix} \qquad H=\begin{bmatrix} h_1\\ h_2\\ h_3\\ \vdots\\ \\ \\ \\ \\ h_{2N-5}\\ h_{2N-4}\end{bmatrix} \tag{24}$$

in which:

$$\begin{aligned}
h_1 &= \frac{20}{T_2^2}(P_1-P_2)+\frac{12}{T_2}v_1+3a_1+\frac{1}{3}T_2 j_1\\
h_2 &= \frac{48}{T_2^3}(P_2-P_1)+\frac{24}{T_3^3}(P_3-P_2)-\frac{30}{T_2^2}v_1-\frac{6}{T_2}a_1\\
h_{2k-1} &= 12(P_{k+1}-P_k), \quad k=2,\dots,N-3\\
h_{2k} &= 8T_{K+2}^3(P_{k+1}-P_k)+4T_{K+1}^3(P_{k+2}-P_{k+1}), \quad k=2,\dots,N-3\\
h_{2N-5} &= \frac{6}{T_{N-1}^2}(P_{N-1}-P_{N-2})\\
h_{2N-4} &= \frac{8}{T_{N-1}^3}(P_{N-1}-P_{N-2})+\frac{10}{T_N^3}(P_N-P_{N-1})-\frac{4}{T_N^2}v_N+\frac{1}{2T_N}a_N
\end{aligned} \tag{25}$$

The "5455" trajectory

The novel "5455" trajectory can be thought as an improved version of trajectory "545". This primitive function is based on the use of 5^{th} order polynomial functions for the first and the last two segments of the trajectory. All the other segments are described by 4^{th} order polynomial functions. In this way, it is possible to achieve the following goals:

1. jerk is continuous for the whole trajectory
2. the value of jerk at the first and last via point can be imposed

It should be pointed out that the "545" trajectory only allows to choose the value of the jerk only at the first via point. Therefore the choice of a "5455" trajectory sports an extra degree of freedom, in comparison to the "545". When planning a "5455" trajectory, the user might choose to achieve a null jerk at the extremities of the path (i.e. $j_0=0$ and $j_N=0$), or he can choose to impose two arbitrary values for j_0 and j_N. The first choice has more appeal when maximum smoothness is the goal, while the second one can be efficiently exploited when the trajectory planning includes a composite trajectory. One example of a profitable use of nonzero initial and final jerk is the planning of trajectories with repetitive tasks. This case will be dealt with at the end of this section.

The formulation of the "5455" is the same as the one developed for the "545", with the exception of the boundary conditions for the two last segment of the trajectory. The 6 boundary conditions for the next-to-last segment are:

$$\begin{aligned}
F_{N-2}(0) &= P_{N-2}\\
v_{N-2}(0) &= v_{N-2}\\
a_{N-2}(0) &= a_{N-2}\\
F_{N-2}(T_{N-1}) &= P_{N-1}\\
v_{N-2}(T_{N-1}) &= v_{N-1}\\
a_{N-2}(T_{N-1}) &= a_{N-1}
\end{aligned} \tag{26}$$

which can be translated into the following matrix expression, after the expression of the 5^{th} order interpolating function is used:

$$\begin{bmatrix} B_{N-2,1}\\ B_{N-2,2}\\ B_{N-2,3}\\ B_{N-2,4}\\ B_{N-2,5}\\ B_{N-2,6}\end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0\\ -\frac{10}{T_{N-1}^3} & -\frac{6}{T_{N-1}^2} & -\frac{3}{2T_{N-1}} & \frac{10}{T_{N-1}^3} & -\frac{4}{T_{N-1}^2} & \frac{1}{2T_{N-1}}\\ \frac{15}{T_{N-1}^4} & \frac{8}{T_{N-1}^3} & \frac{3}{2T_{N-1}^2} & -\frac{15}{T_{N-1}^4} & \frac{7}{T_{N-1}^3} & -\frac{1}{T_{N-1}^2}\\ -\frac{6}{T_{N-1}^5} & -\frac{3}{T_{N-1}^4} & -\frac{1}{2T_{N-1}^3} & \frac{6}{T_{N-1}^5} & -\frac{3}{T_{N-1}^4} & \frac{1}{2T_{N-1}^3}\end{bmatrix}\cdot\begin{bmatrix} P_{N-2}\\ v_{N-2}\\ a_{N-2}\\ P_{N-1}\\ v_{N-1}\\ a_{N-1}\end{bmatrix} \tag{27}$$

Boundary conditions for the last segment of the trajectory are:

$$
\begin{aligned}
F_{N-1}(0) &= P_{N-1}\\
v_{N-1}(0) &= v_{N-1}\\
F_{N-1}(T_N) &= P_N\\
v_{N-1}(T_N) &= v_N\\
a_{N-1}(T_N) &= a_N\\
j_{N-1}(T_N) &= j_N
\end{aligned} \tag{28}
$$

Notice that here, unlike the ”545”, we can impose the jerk even at the last via-point, i.e. j_N. This is possible because the use of 5^{th} order polynomial function for a single segment instead of a 4^{th} order, as in the ”545”, allows to use one more degree of freedom on the choice of the boundary conditions. Using the same procedure as before, boundary conditions for the last segment can be condensed in a single equation:

$$
\begin{bmatrix} B_{N-1,1}\\ B_{N-1,2}\\ B_{N-1,3}\\ B_{N-1,4}\\ B_{N-1,5}\\ B_{N-1,6} \end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
-\frac{10}{T_N^2} & -\frac{4}{T_N} & \frac{10}{T_N^2} & -\frac{6}{T_N} & \frac{3}{2} & -\frac{1}{6T_N}\\
\frac{20}{T_N^3} & \frac{6}{T_N^2} & -\frac{20}{T_N^3} & \frac{14}{T_N^2} & -\frac{4}{T_N} & \frac{1}{2}\\
-\frac{15}{T_N^4} & -\frac{4}{T_N^3} & \frac{15}{T_N^4} & -\frac{11}{T_N^3} & \frac{7}{2T_N^2} & -\frac{1}{2T_N}\\
\frac{4}{T_N^5} & \frac{1}{T_N^4} & -\frac{4}{T_N^5} & \frac{3}{T_N^4} & -\frac{1}{T_N^3} & \frac{1}{6T_N^2}
\end{bmatrix} \cdot
\begin{bmatrix} P_{N-1}\\ v_{N-1}\\ P_N\\ v_N\\ a_N\\ j_N \end{bmatrix} \tag{29}
$$

The new formulations of matrices **D** and **H** are the same as in Eq. 24, with the exception of 4 elements of matrix **H**:

$$
\begin{aligned}
h_{2N-7} &= \frac{6}{T_{N-2}^2}(P_{N-2}-P_{N-3})\\
h_{2N-6} &= \frac{8}{T_{N-2}^3}(P_{N-2}-P_{N-3}) + \frac{10}{T_{N-1}^3}(P_{N-1}-P_{N-2})\\
h_{2N-5} &= \frac{20}{T_N^2}(P_N-P_{N-1}) - \frac{12}{T_N}v_N + 3a_N - \frac{1}{3}T_N j_N\\
h_{2N-4} &= \frac{20}{T_{N-1}^3}P_{N-2} - \left(\frac{20}{T_{N-1}^3} - \frac{40}{T_N^3}\right)P_{N-1} - \frac{40}{T_N^3}P_N +\\
&\quad + \frac{84}{3T_N^2}v_N - \frac{8}{T_N}a_N + j_N
\end{aligned} \tag{30}
$$

The above conditions can be used to compute the new correct formulation for matrix **M**, which is reported in Appendix A.

Trajectory planning for repetitive tasks

As stated before, the proposed ”5455” method allows to impose the value of jerk at the initial and final point of the trajectory. We here exploit this feature to efficiently plan trajectories which include repetitive tasks. Let us take into consideration a task which includes: 1) an initial (or ”speed-up”) phase 2) a repetitive task which is to be repeated M time 3) a final (or ”slow-down”) phase. Each phase is described by a set of via-points, with the last via-point of phase 1 equal to the first via-point of phase 2. The same applies for the last point of phase 2 and the first of phase 3. If the number of cycles M is large (10 or even more), the numerical solution of the optimization algorithms can be jeopardized or, in the best option, it can be dramatically slowed down. A possible solution is to adopt the following strategy, which sports a lower computational burden: we can split the problem into three parts, to be solved in the specified order:

1. determine through an optimization problem the optimal values of the time intervals between each two consecutive via-points belonging to the cycle to be repeated, i.e. $[T_{k,1}, T_{k,2}, \dots, T_{k,N}]$, and of the initial and final conditions $[IC_k = FC_k]$
2. determine the optimal trajectory for the phase 1, with zero initial conditions (on velocity, acceleration and jerk), and with final conditions equal to the initial conditions evaluated at the previous step, i.e. IC_k
3. determine the optimal trajectory for the phase 3, with zero final conditions (on velocity, acceleration and jerk), and with initial conditions equal to the final conditions evaluated at the first step, i.e. FC_k

In this way we have a procedure that guarantees the continuity of jerk for all the duration of the task, and that starts and ends with null jerk for minimum induced vibrations.

Experimental results: comparison with other methods

In this section the effectiveness of two novel trajectory planning algorithm is evaluated trough experimental comparison with other common methods available in literature. For all the test cases, minimum time trajectory with bounded speed, acceleration and jerk are taken into consideration. Results are obtained using a 3 d.o.f. Cartesian manipulator, which uses commonly available drive units. A picture of the robot is available in Fig. 1. Therefore it replicates the structure and the hardware of commonly available industrial robots. The kinematic constraints have been chosen as in Table 1, according to the datasheet values of the robot prototype. The maximum value of jerk is an arbitrary value, since this data is not available from any datasheet. Generally this limit can be used to control the smoothness of the trajectory.

TABLE 1. KINEMATIC CONSTRAINTS

	Min value	Max value
Velocity	-225 mm/s	225 mm/s
Acceleration	-2400 mm/s^2	2400 mm/s^2
Jerk	-2400 mm/s^3	2400 mm/s^3

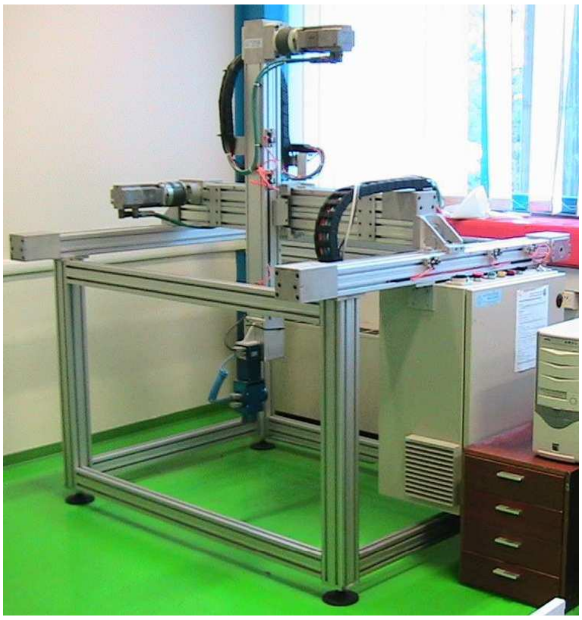

FIGURE 1. THE CARTESIAN MANIPULATOR USED FOR EXPERIMENTAL TESTS

Pick & place trajectory

This tests is aimed at evaluating the accelerations along the Z axis on the end effector on the manipulator for a pick & place trajectory. Results are provided for several planning algorithms, such as the SPL3J [5], the ”434” [6], ”445” [7], ”5455” and BSPL5J [5]. Simulated accelerations are shown in Fig. 2. The trajectory is specified as a sequence of 8 via-points. As it can be seen in Fig. 2, the SPL3J methods produces an interpolation which is quite prone to induce vibrations on the end effector, since there are 10 points with discontinuous jerk. The application of the ”434” and the ”445” algorithms results in trajectories with 8 and 2 jerk discontinuities, respectively. Null initial and final jerk can be obtained only by the ”5455” and the BSPL5J methods. In the case under consideration, it can be seen that the SPL3J, the ”434” and the ”445” methods produce similar results, since there is a small variation among them in terms of total execution time, and in the magnitude of the peak acceleration. As it can be seen in Fig. 3, and as expected from theoretical results, the ”455” methods can produce the lowest amount of acceleration among the three algorithms considered above. The trajectories planned with ”5455” and BSPL5J methods have virtually the same execution time, while the peak value of the simulated acceleration (see Fig. 2) for the BSPL5J is the lowest among the 5 methods. On the other hand, the experimental results shows that the measured acceleration on the end effector of the robot for the case of the ”5455” is not worse than the one obtained with the BSPL5J method. The exact values of total execution time, of root mean square (RMS) acceleration and of the time integral of squared acceleration measured on the end-effector are reported in Table 2 to provide a more accurate evaluation of the experimental evidences.

TABLE 2. EXPERIMENTAL TESTS: PICK & PLACE TRAJECTORIES

	SPL3J	434	455	5455	BSPL5J
execution time $[s]$	4.838	5.002	4.888	5.671	5.660
RMS acc. $[m/s^2]$	0.2229	0.2026	0.2052	0.1520	0.1569
$\int acc^2$ $[m^2/s^3]$	0.4146	0.3974	0.3912	0.2595	0.2564

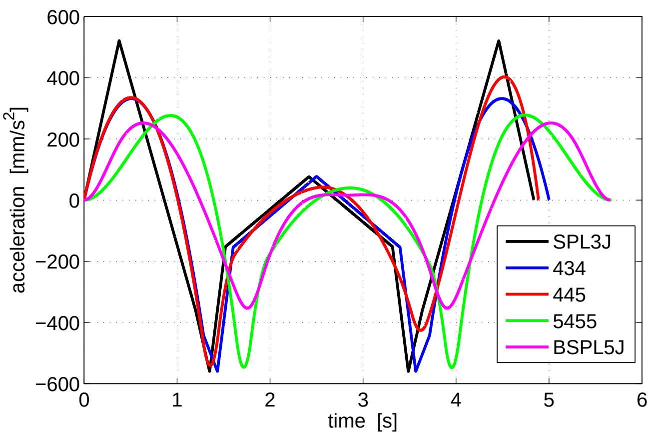

FIGURE 2. COMPUTED ACCELERATIONS ALONG THE Z AXIS FOR THE PICK & PLACE TRAJECTORY

Circular trajectory

This tests involves planning a trajectory by interpolating a sequence of 7 via points located on a circle which lies on a plane that is not parallel to the XY plane. In this way all the three axes of the robot are moved at the same time. Figure 4 shows the planned accelerations for the Z axis of the robot. The analysis of the figure shows that all the methods under consideration pro-

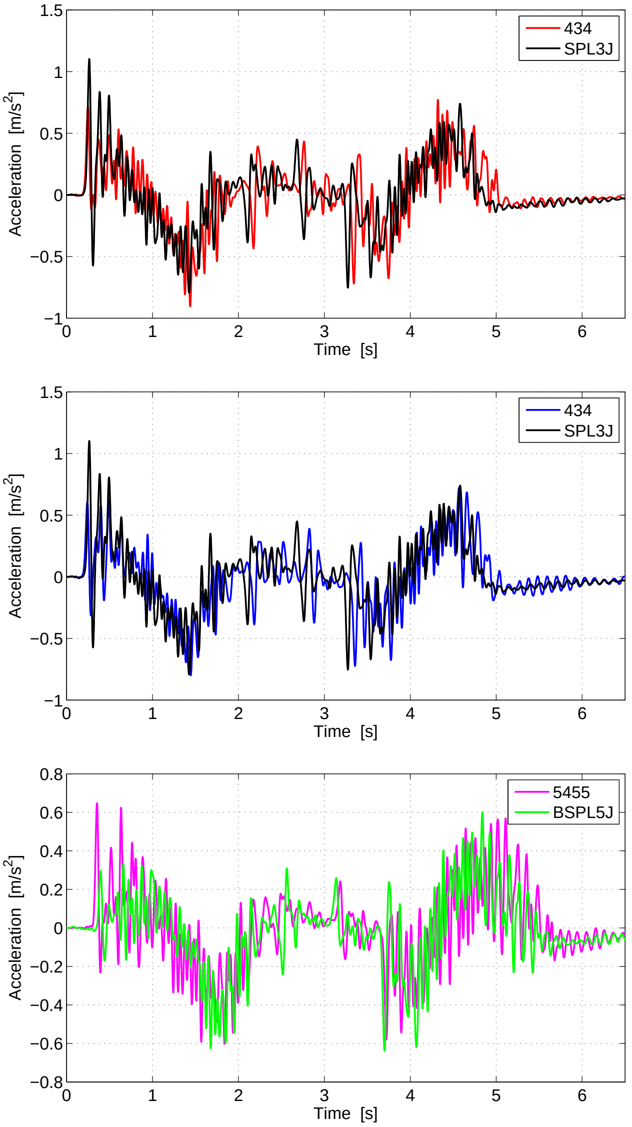

FIGURE 3. MEASURED ACCELERATIONS ALONG THE Z AXIS FOR THE PICK & PLACE TRAJECTORY

duce different total execution time. The fastest trajectory is, as expected, the SPL3J, while the slowest is the "5455". For the exact values, refer to Table 3. From the data available in Fig. 5 and Table 3, it can be seen that the BSPL5J produces the trajectory with the lowest time integral of the squared acceleration on the end-effector, while the "5455" achieves the lowest value of RMS acceleration. Therefore we can evaluate that the level of performance in terms of vibration for the "5455" and the BSPL5J are basically the same.

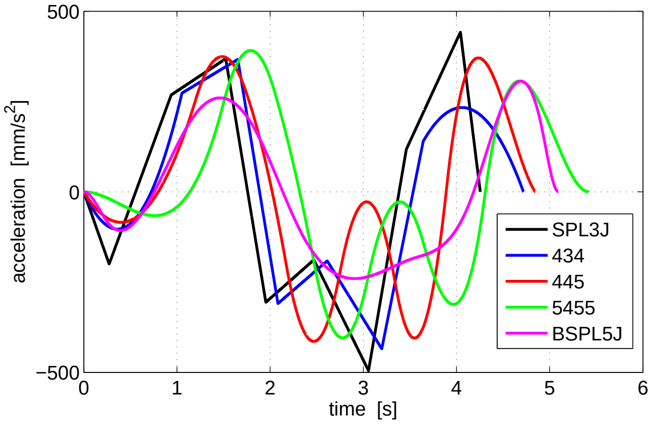

FIGURE 4. COMPUTED ACCELERATIONS ALONG THE Z AXIS FOR THE CIRCULAR TRAJECTORY

TABLE 3. EXPERIMENTAL TESTS: CIRCULAR TRAJECTORIES

	SPL3J	434	455	5455	BSPL5J
execution time $[s]$	4.255	4.718	4.844	5.4190	5.095
RMS acc. $[m/s^2]$	0.2424	0.2218	0.1999	0.1870	0.1942
$\int acc^2$ $[m^2/s^3]$	0.5022	0.439	0.3708	0.3610	0.3455

CONCLUSION

In this paper a new trajectory planning algorithm has been presented. It allows to interpolate a set of N via-points obtaining the continuity of jerk along the whole path. Arbitrary constraints can be imposed on velocity, acceleration and jerk of each joint of the robot independently, thus allowing the use of the proposed method for virtually every industrial manipulator. Moreover the choice of interpolating functions allows for the user to impose an arbitrary value of jerk at the first and the last via-point. This feature is used to plan trajectories which involve repetitive tasks in a very efficient manner. Both numerical and experimental results are presented to show the efficiency of the proposed solution. Comparison with other trajectory planning algorithms prove that the proposed approaches can achieve a good compromise between the level of smoothness and the total execution time, when minimal time trajectories with bounded speed, acceleration and jerk are taken into account.

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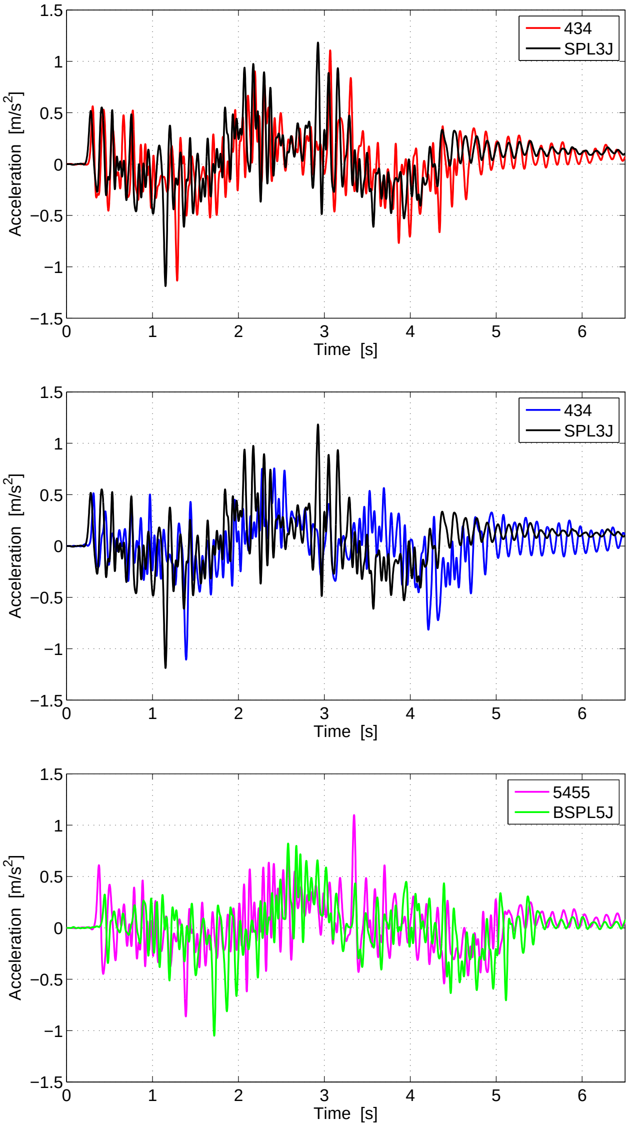

FIGURE 5. MEASURED ACCELERATIONS ALONG THE Z AXIS FOR THE CIRCULAR TRAJECTORY

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Appendix A: Matrix M for the ”545” and ”5455” trajectories

$$
\mathrm{M}_{545} = \begin{bmatrix}
-\frac{8}{T_2} & 1 & 0 & 0 & \dots & & & \dots & 0 & 0 \\
-\frac{36}{T_2^2}-\frac{18}{T_3^2} & -\frac{6}{T_3} & -\frac{6}{T_3^2} & 0 & 0 & \dots & & & & 0 \\
6T_3 & T_3^2 & 6T_3 & -T_3^2 & 0 & 0 & \dots & & & \vdots \\
5T_3T_4^3 & T_3^2T_4^3 & 3T_3T_4^3+3T_3^3T_4 & T_3^3T_4^2 & T_3^3T_4 & 0 & 0 & & & \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
\vdots & \vdots & & & & & & & & \\
 & & & 0 & 6T_{N-2} & T_{N-2}^2 & 6T_{N-2} & -T_{N-2}^2 & 0 & 0 \\
 & & & 0 & 5T_{N-2}T_{N-1}^3 & T_{N-2}^2T_{N-1}^3 & 3T_{N-2}T_{N-1}^3+3T_{N-2}^3T_{N-1} & T_{N-2}^3T_{N-1}^2 & T_{N-2}^3T_{N-1} & 0 \\
0 & & & & & 0 & \frac{3}{T_{N-1}} & \frac{1}{2} & \frac{3}{T_{N-1}} & -\frac{1}{2} \\
0 & 0 & \dots & & & 0 & \frac{5}{T_{N-1}^2} & \frac{1}{T_{N-1}} & \frac{3}{T_{N-1}^2}+\frac{6}{T_N^2} & \frac{3}{2T_N}
\end{bmatrix}
$$

$$
M_{5455} = \begin{bmatrix}
-\frac{8}{T_2} & 1 & 0 & 0 & \dots & & & & & \dots & 0 & 0 \\
-\frac{36}{T_2^2}-\frac{18}{T_3^2} & -\frac{6}{T_3} & -\frac{6}{T_3^2} & 0 & 0 & \dots & & & & & & 0 \\
6T_3 & T_3^2 & 6T_3 & -T_3^2 & 0 & 0 & \dots & & & & & \vdots \\
5T_3T_4^3 & T_3^2T_4^3 & 3T_3T_4^3+3T_3^3T_4 & T_3^3T_4^2 & T_3^3T_4 & 0 & 0 & & & & & \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
\vdots & \vdots & & & & & & & & & & \\
 & & & 0 & 6T_{N-3} & T_{N-3}^2 & 6T_{N-3} & -T_{N-3}^2 & 0 & 0 & 0 & 0 \\
 & & & 0 & 5T_{N-3}T_{N-2}^3 & T_{N-3}^2T_{N-2}^3 & 3T_{N-3}T_{N-2}^3+3T_{N-3}^3T_{N-2} & T_{N-3}^3T_{N-2}^2 & T_{N-3}^3T_{N-2} & 0 & 0 & 0 \\
0 & & \dots & & & 0 & \frac{3}{T_{N-2}} & \frac{1}{2} & \frac{3}{T_{N-2}} & -\frac{1}{2} & 0 & 0 \\
0 & & \dots & & & 0 & \frac{5}{T_{N-2}^2} & \frac{1}{T_{N-2}} & \frac{3}{T_{N-2}^2}+\frac{6}{T_{N-1}^2} & \frac{3}{2T_{N-1}} & \frac{4}{T_{N-1}^2} & -\frac{1}{2T_{N-1}} \\
0 & & \dots & & & 0 & 0 & 0 & 0 & 0 & \frac{8}{T_N} & 1 \\
0 & & \dots & & & 0 & 0 & 0 & -\frac{8}{T_{N-1}^2} & -\frac{1}{T_{N-1}} & -\frac{12}{T_{N-1}^2}-\frac{12}{T_N^2} & \frac{3}{T_{N-1}}
\end{bmatrix}
$$