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KINEMATIC AND DYNAMIC ANALYSIS OF FLEXIBLE-LINK PARALLEL ROBOTS BY MEANS OF AN ERLS APPROACH

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ABSTRACT

The use of parallel kinematic structures allows to design light manipulators with higher dynamic performance with respect to serial robots,. In this work, the issue of accurately modelling the dynamics of lightweight flexible-link parallel robots has been investigated. The Equivalent Rigid-Link System (ERLS) formulation, useful for describing the dynamic evolution of 3D serial robots with flexible-links, has been extended to Parallel-Kinematic-Machines (PKM) both from the theoretical and from the software implementation points of view. Standard robotics concepts of 3D kinematics are exploited to formulate and solve the ERLS kinematics, thus allowing to easily work with this formulation. A simulator, capable of predicting the deformations due to the elasticity of the links, has been developed and some industrial case-studies have been implemented to validate it. The formulation and the developed simulator will give the opportunity to extensively study the deformations of parallel manipulators, allow to predict the vibrations of the system, and, then, compensate them.

INTRODUCTION

Flexible-link manipulators and parallel robots are two of the main components and interesting topics of robotics.

The former, the flexible-link manipulators, are addressed due to its light weight and higher acceleration motion with respect to rigid-link systems, requirements of the modern industry. The latter, the parallel manipulators [1], allow to improve dynamic performance of robots since lower inertia, higher stiffness and accuracy are achievable with respect to serial robots. Thus, taking into account the growing push towards high precision and high efficiency, the flexible-link parallel robots can be considered a new hot topic in robotics research. On this regard it has to be said that, since parallel manipulators consist of multiple closed-loop chains, both their analysis and dynamic modeling are more complex than in serial mechanisms and, as a conse-

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quence, parallel manipulators with flexible-link dynamic problems have not been yet extensively studied. Several researchers have focused their investigations on parallel robots, but most of them are restricted to the rigid parallel robots. Few works deal with both flexible-links and parallel robots (e.g. [2]). A lot of work have been done during the last decades on the definition of accurate mathematical models for flexible systems (also working at high speeds), from the first work on elastic linkages by Neubauer, Cohen and Hall [3]. In this first approach the mechanism elastic links are treated as continuous systems possessing infinite degrees-of-freedom. Some years later, starting from the work of Winfrey [4] and Imam [5], mechanism elastic links have been represented as discrete systems possessing finite elastic degrees of freedom. Such models have been developed making use of discretization methods like the finite element method (FEM). To predict the elastic dynamic behavior of flexible serial robots and flexible single closed-chain planar mechanisms, a number of approaches have been developed (e.g. [6–9]).

Recently, more attention has been given to parallel robots and an efficient method to enable designers and roboticists to predict and control the elastic dynamic behavior of parallel robots with flexible-links is still an issue.

The method here considered and proposed is based on the theory proposed by Giovagnoni [10] and is valid for any chain of flexible bodies. It is developed making use of the Equivalent Rigid-Link System (ERLS) proposed by Chang and Hamilton [11]. This approach allows to decouple the kinematic equations of the ERLS from the compatibility equations of the displacements at the joints without neglecting the mutual influence between rigid body motion and vibration. On the contrary, in other formulations such as the Floating Frame of Reference (FFR) formulation [8], the connections between different deformable bodies, are defined in the global coordinate system and the resulting constraint equations are coupled, i.e. rigid and flexible terms are mixed, and do not have an immediate and easy formulation. As a consequence, one of the main advantages of the approach is that the standard concepts of 3-D kinematics can be adopted to formulate and solve the ERLS kinematics. Recently, the ERLS formulation has been extended for modeling any chain of spatial serial robots [12].

The next sections will explain the mathematical ERLS model giving particular stress on the modifications adopted for taking into account the possibility to describe multi closed-loop robots, i.e. general parallel robots. Then, the implemented simulator will be described and some test-cases will be considered and treated in order to show the effectiveness of the method.

The ERLS 3D-model for flexible link manipulators

Kinematic definitions

In the proposed model an Equivalent Rigid-Link Mechanism is defined so that the elastic displacements with respect to it can

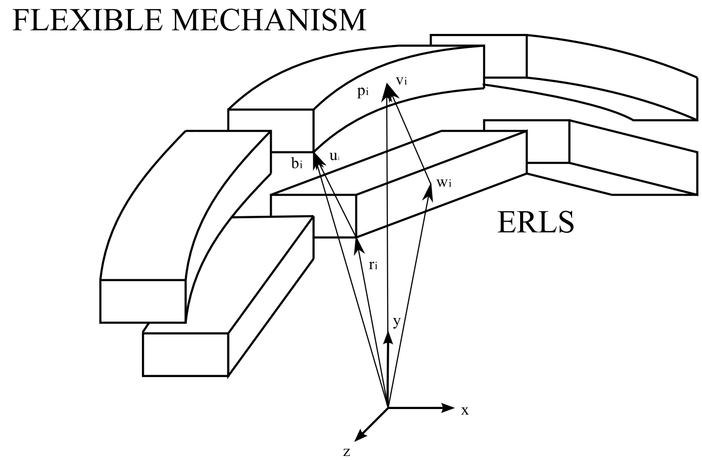

FIGURE 1. Definitions for the ERLS model

be considered. The ERLS follows the kinematic laws for a rigid-link mechanism and each link is subdivided, in this case, into spatial beam finite elements modeled with the Euler-Bernoulli theory. The following definitions are posed according with fig.1 and with respect to a fixed reference frame (x, y, z):

- $\mathbf{u}_i$: nodal elastic displacement vector for the i-th finite element;
- $\mathbf{r}_i$: nodal position vector for the i-th element of the ERLS;
- $\mathbf{b}_i$: nodal position vector for the i-th finite element;
- $\mathbf{v}_i$: elastic displacement vector of a generic point inside the i-th element;
- $\mathbf{w}_i$: position vector of the generic point of the i-th element of the ERLS;
- $\mathbf{p}_i$: position vector of the generic point of the i-th finite element;

the previous definitions are linked each other by:

$$\mathbf{b}_i = \mathbf{r}_i + \mathbf{u}_i \tag{1}$$

$$\mathbf{p}_i = \mathbf{v}_i + \mathbf{w}_i \tag{2}$$

In order to apply the virtual work principle, the expressions of the virtual displacements and of the acceleration of a generic point inside the i-th finite element will be computed (for more details see [12]).

Dynamic relations

The dynamic equations of motion for the flexible mechanism can be obtained by applying the principle of virtual work:

$$\delta W^{inertia} + \delta W^{elastic} = -\delta W^{external} \tag{3}$$

that can be exploited in the following:

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$$\sum_i \int_{v_i} \delta \mathbf{p}_i^T \ddot{\mathbf{p}}_i \rho_i dv + \sum_i \int_{v_i} \delta \varepsilon_i^T \mathbf{D}_i \varepsilon_i dv = \sum_i \int_{v_i} \delta \mathbf{p}_i^T \mathbf{g} \rho_i dv + (\delta \mathbf{u}^T + \delta \mathbf{r}^T)\mathbf{f} \tag{4}$$

where $\mathbf{D}_i$ is the stress-strain matrix, ε_i the strain vector and ρ_i the mass density for the i-th element, $\mathbf{g}$ is the gravity acceleration vector and $\mathbf{f}$ is the vector of the concentrated external forces and torques. The total virtual work is split into the integrals over element volumes v_i and in the virtual work due to $\mathbf{f}$; $\delta\mathbf{u}$ and $\delta\mathbf{r}$ refer to all the nodes of the model.

Constraints for general parallel robots The constraints due to the kinematic pairs, i.e. the formulation of the compatibility equations, depend on the frame in which the nodal elastic displacements are expressed. For this issue, in addition to the previously defined (x,y,z) fixed reference frame, a frame (x_i,y_i,z_i) that follows the ERLS motion is defined. This local reference frame can be expressed with respect to the global one by means of a set of free coordinates $\mathbf{q}$, i.e. the rigid degrees of freedom, defined to describe the mobility of the system. In order to correctly account for the displacement interpolations inside the finite elements, a local-to-global transformation matrix $\mathbf{R}_i(\mathbf{q})$ and a block-diagonal rotation matrix $\mathbf{T}_k^i(\mathbf{q})$, expressing the transformation from the frame k, in which are expressed the nodal elastic displacements of the i-th finite element $\mathbf{u}_i^k$, to the local reference frame, are defined.

Two consecutive links have different local frames, that in a common serial robots are fixed according to the Denavit-Hartenberg (DH) notation but, in parallel ones, it can be a non easy task since, by applying DH procedure, some ambiguities have to be solved for this kind of robots. Till now, there is not a commonly accepted notation and some contributions have been presented in the literature like the graph based representation (e.g. [13]). What is surely better to choose for a good readability, is to take as the first links those that are driven, i.e. the rigid degrees of freedom. Anyway, once a notation is fixed for the mechanism, the compatibility equation that describe the constraint of the robots have to be posed.

In serial robots it's enough to ensure that the last node of the i-th link and those of the first node of the $(i{+}1)$-th link must be expressed in the same local frame. Within a kinematic chain, for all the beam elements of a link except the last (i.e. the one connected to the following link), $\mathbf{T}_{k,i}^i$ is a block-diagonal identity matrix, because the suitable reference frame k for expressing the elastic displacements $\mathbf{u}_i^k$ of each node of the beam element coincides with the local link frame *i.* On the other hand, the elastic displacements of the last node of the i-th link have to be rotated into the local frame of the $(i{+}1)$-th link, so that the kinematic entities are defined with respect to the same reference system.

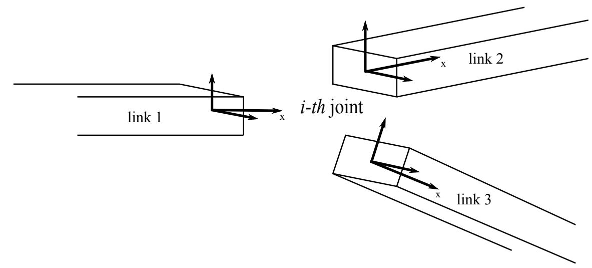

FIGURE 2. An example for the connection notation chosen: link 1 is ”ending” to the i-th node, while links 2 and 3 are ”starting” from the same node. As a consequence, the reference frame chosen for the i-th node will be the second link's one

In parallel robots the foregoing approach is not anymore well-defined since it's not clear which one is the ”previous” and the ”next” link in a closed-loop chain. The task of find a common reference frame for the common node for two consecutive links can be stated as the task of find a reference for each joint: this reference has to be adopted for all the nodes of the links that converge on that node (whilst the other nodes can continue to use the local-link reference). The following rule has been chosen:

- for each link a local reference (x_k,y_k,z_k) whose x-axis is aligned along the link longitudinal axis is set and for each joint a progressive number is chosen (since, at the time of writing, there is not a commonly accepted unified notation for the whole serial, parallel and hybrid mechanisms, for instance the notation in [13] can be adopted) A link, with respect to its starting/ending node, is defined as ”starting” from a node if the x-axis of its reference frame is coming out from that node, otherwise it's defined as ”ending” to a node. Look at the fig.2 for an example.
- for each joint the reference frame to be chosen is:
  - the reference frame of the link with lower index, among those ”starting” from that i-th joint, if there's at least one link ”starting” from that joint. Otherwise:
  - the reference frame of the link with lower index among those links ”ending” to that joint.

This proposed rule has the advantage of being consistent with the previous rule used for serial robots, since it is just an extension for manipulators where more than two links can converge on the same joint.

Moreover, a kind of rule has to be stated in order to say how to enumerate the whole degrees of freedom for the robots. In serial robots this task was straightforward: just put in order the link from the base-frame one to the end-effector, then put in order the degrees of freedom inside of each link, and when it's necessary just ”overlap” those degrees of freedom that are in common for two consecutive link. In parallel manipulator this is not any more straightforward, since the closed loop-chain can

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be more than one and the degrees of freedom to be "overlapped" can be among more than two nodes. The following rule has been chosen:

- take one link at a time, respecting their increasing order number
- the first degrees of freedom are just the degrees of freedom of the first link, in order
- for each of the following links (taken in order, according to the first item): take its degrees of freedom in order; if the under analysis degree of freedom has to been overlapped to one just previously written (i.e. it's in common with a degree of freedom of a previous link) put it in the previously chosen position, otherwise add it sequentially in the column of the degrees of freedom.

This rule has the advantage to be back-compatible with the ERLS-based formulation and implemented algorithms for the serial robots, when a serial robot is taken.

With such a system, the compatibility equations at the joints are written and included considering only the elastic displacements and are never used explicitly, since they are automatically taken into account when assembling the system matrices, thus avoiding the need to write a set of nonlinear algebraic constraints equations. That is one of the main advantages of the ERLS method.

Local nodal equilibrium From (4) two different set of nodal elastic virtual displacements $\delta\mathbf{u}_i^k$ and virtual displacements of the ERLS $\delta\mathbf{r}_i^k$ can be obtained, and they are completely independent.

The local nodal equilibrium equations, can be obtained from (4) by considering: $\delta\mathbf{r}_i = 0$ and $\delta\mathbf{u}_i \neq 0$. In order to do that a more detailed expression for $\delta\mathbf{p}_i$ and $\ddot{\mathbf{p}}_i$ need to be found. In order to do that, the (2) with virtual displacements is used: the first term on the right hand side expressed by means of virtual displacements, the interpolation function matrix $\mathbf{N}_i(x_i, y_i, z_i)$ can be used to interpolate infinitesimal rigid-body displacements, while for the second term both virtual nodal elastic displacements $\delta\mathbf{u}_i$ and virtual displacements $\delta\mathbf{q}$ of the generalized coordinates have to be considered. That is, with virtual displacement , see (2), is simply:

$$\delta\mathbf{p}_i = \delta\mathbf{w}_i + \delta\mathbf{v}_i \tag{5}$$

where the first addend $\delta\mathbf{w}_i$ can be written by means of virtual displacement $\delta\mathbf{r}_i$ in the following way:

$$\delta\mathbf{w}_i = \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_i(\mathbf{q})\delta\mathbf{r}_i \tag{6}$$

when the second addend $\delta\mathbf{v}_i$ can be expressed by means of nodal virtual displacement $\delta\mathbf{u}_i$ and ERLS virtual displacement through generalized coordinates displacements $\delta\mathbf{q}$:

$$\begin{aligned}\delta\mathbf{v}_i = {} & \delta\mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_i(\mathbf{q})\mathbf{u}_i \\ & + \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\delta\mathbf{T}_i(\mathbf{q})\mathbf{u}_i \\ & + \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_i(\mathbf{q})\delta\mathbf{u}_i\end{aligned} \tag{7}$$

So the expression for the displacement $\delta\mathbf{p}_i$ is:

$$\begin{aligned}\delta\mathbf{p}_i = {} & \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q})\delta\mathbf{r}_i^k \\ & + \delta\mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q})\mathbf{u}_i^k \\ & + \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\delta\mathbf{T}_k^i(\mathbf{q})\mathbf{u}_i^k \\ & + \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q})\delta\mathbf{u}_i^k\end{aligned} \tag{8}$$

that in the assumptions to get the local nodal equilibrium $\delta\mathbf{r}_i = 0$; $\delta\mathbf{R}_i = 0$; $\delta\mathbf{T}_i = 0$; $\delta\mathbf{u}_i \neq 0$; becomes simply:

$$\delta\mathbf{p}_i = \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q})\delta\mathbf{u}_i^k \tag{9}$$

For the accelerations $\ddot{\mathbf{p}}_i$, it's straightforward to write the following from (2):

$$\ddot{\mathbf{p}}_i = \ddot{\mathbf{w}}_i + \ddot{\mathbf{v}}_i \tag{10}$$

where $\ddot{\mathbf{w}}_i$, looking at (6) can be expressed as:

$$\ddot{\mathbf{w}}_i = \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_i(\mathbf{q})\ddot{\mathbf{r}}_i \tag{11}$$

whilst $\ddot{\mathbf{v}}_i$, looking at (7), can be written as:

$$\begin{aligned}\ddot{\mathbf{v}}_i = {} & \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_i(\mathbf{q})\ddot{\mathbf{u}}_i \\ & + 2(\dot{\mathbf{R}}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_i(\mathbf{q}) + \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\dot{\mathbf{T}}_i(\mathbf{q}))\dot{\mathbf{u}}_i \\ & + (\ddot{\mathbf{R}}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_i(\mathbf{q}) + 2\dot{\mathbf{R}}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\dot{\mathbf{T}}_i(\mathbf{q}) \\ & + \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\ddot{\mathbf{T}}i(\mathbf{q}))\mathbf{u}_i\end{aligned} \tag{12}$$

so, the final expression for $\ddot{\mathbf{p}}_i$ is

$$\begin{aligned}\ddot{\mathbf{p}}_i = {} & \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q})\ddot{\mathbf{r}}_i^k + \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q})\ddot{\mathbf{u}}_i^k \\ & + 2(\dot{\mathbf{R}}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q}) + \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\dot{\mathbf{T}}_k^i(\mathbf{q}))\dot{\mathbf{u}}_i^k \\ & + (\ddot{\mathbf{R}}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q}) + 2\dot{\mathbf{R}}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\dot{\mathbf{T}}_k^i(\mathbf{q}) \\ & + \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\ddot{\mathbf{T}}_k^i(\mathbf{q}))\mathbf{u}_i^k\end{aligned} \tag{13}$$

Hence, by substituting the simplified expression for $\delta\mathbf{p}_i$ in (9) and the last expression of $\ddot{\mathbf{p}}_i$ (13) in the dynamic equation (4) it holds:

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$$\sum_i \int_{v_i} [\delta \mathbf{u}_i^{k\,T} \mathbf{T}_{k,i}^{i\,T} \mathbf{N}_i^T \mathbf{R}_i^T][\mathbf{R}_i \mathbf{N}_i \mathbf{T}_{k,i}^i \ddot{\mathbf{r}}_i^k + \mathbf{R}_i \mathbf{N}_i \mathbf{T}_{k,i}^i \ddot{\mathbf{u}}_i^k$$
$$+ 2(\dot{\mathbf{R}}_i \mathbf{N}_i \mathbf{T}_{k,i}^i + \mathbf{R}_i \mathbf{N}_i \dot{\mathbf{T}}_{k,i}^i)\dot{\mathbf{u}}_i$$
$$+ (\ddot{\mathbf{R}}_i \mathbf{N}_i \mathbf{T}_k^i + 2\dot{\mathbf{R}}_i \mathbf{N}_i \dot{\mathbf{T}}_k^i + \mathbf{R}_i \mathbf{N}_i \ddot{\mathbf{T}}_k^i)\mathbf{u}_i^k]\rho_i dv$$
$$+ \sum_i \int_{v_i} (\delta \mathbf{u}_k^{i\,T} \mathbf{T}_{k,i}^{iT} \mathbf{B}_i^T)\mathbf{D}_i \mathbf{B}_i \mathbf{T}_{k,i}^i \mathbf{u}_i^k dv$$
$$= \sum_i \int_{v_i} (\delta \mathbf{u}_i^{kT} \mathbf{T}_{k,i}^{iT} \mathbf{N}_i^T \mathbf{R}_i^T)\mathbf{g}\rho_i dv + (\delta \mathbf{u}^T + \delta \mathbf{r}^T)\mathbf{f} \tag{14}$$

In the (14) the mass, Coriolis, gyroscopic damping, centrifugal stiffness and stiffness contributions can be summarized by posing:

$$\mathbf{M}_i = \int_{v_i} \mathbf{T}_{k,i}^{i\,T} \mathbf{N}_i^T \mathbf{R}_i^T \mathbf{R}_i \mathbf{N}_i \mathbf{T}_{k,i}^i \rho_i dv \tag{15}$$
$$\mathbf{K}_i = \int_{v_i} \mathbf{T}_{k,i}^{iT} \mathbf{B}_i^T \mathbf{D}_i \mathbf{B}_i \mathbf{T}_{k,i}^i dv \tag{16}$$
$$\mathbf{M}_{G1i} = \int_{v_i} \dot{\mathbf{R}}_i \mathbf{N}_i \mathbf{T}_{k,i}^i \rho_i dv \tag{17}$$
$$\mathbf{M}_{G2i} = \int_{v_i} \mathbf{R}_i \mathbf{N}_i \dot{\mathbf{T}}_{k,i}^i \rho_i dv \tag{18}$$
$$\mathbf{M}_{C1i} = \int_{v_i} \ddot{\mathbf{R}}_i \mathbf{N}_i \mathbf{T}_k^i \rho_i dv \tag{19}$$
$$\mathbf{M}_{C2i} = \int_{v_i} \dot{\mathbf{R}}_i \mathbf{N}_i \dot{\mathbf{T}}_k^i \rho_i dv \tag{20}$$
$$\mathbf{M}_{C3i} = \int_{v_i} \mathbf{R}_i \mathbf{N}_i \ddot{\mathbf{T}}_k^i \rho_i dv \tag{21}$$
$$\mathbf{f}_{gi} = \int_{v_i} \mathbf{T}_{k,i}^{iT} \mathbf{N}_i^T \mathbf{R}_i^T \mathbf{g}\rho_i dv \tag{22}$$

Hence, the local nodal equilibrium can be written in the following compact expression:

$$\sum_i \delta \mathbf{u}_i^{kT} \mathbf{M}_i (\ddot{\mathbf{r}}_i + \ddot{\mathbf{u}}_i) + 2\sum_i \delta \mathbf{u}_i^{kT} (\mathbf{M}_{G1i} + \mathbf{M}_{G2i})\dot{\mathbf{u}}_i^k$$
$$+ \sum_i \delta \mathbf{u}_i^{kT} (\mathbf{M}_{C1i} + 2\mathbf{M}_{C2i} + \mathbf{M}_{C3i})\mathbf{u}_i^k + \sum_i \delta \mathbf{u}_i^{kT} \mathbf{K}_i \mathbf{u}_i^k$$
$$= \sum_i \delta \mathbf{u}_i^T \mathbf{f}_{gi} + \delta \mathbf{u}^T \mathbf{f} \tag{23}$$

Global equilibrium A second set of equilibrium equations, the global equilibrium, comes again from the dynamic relation (4). Looking at (8) by considering $\delta q_j \neq 0, j = 1,...,n; \delta \mathbf{u}_i = \mathbf{0}$. Under this assumptions if $\delta \mathbf{R}_i$, $\delta \mathbf{T}_i$ and $\delta \mathbf{r}_i$ terms are expressed as:

$$\delta \mathbf{R}_i = \sum_j (\partial \mathbf{R}_i / \partial q_j)\delta q_j = \mathbf{R}_i' \delta \mathbf{q} \neq \mathbf{0} \tag{24}$$
$$\delta \mathbf{T}_i = \sum_j (\partial \mathbf{T}_i / \partial q_j)\delta q_j = \mathbf{T}_i' \delta \mathbf{q} \neq \mathbf{0} \tag{25}$$
$$\delta \mathbf{r}_i = \sum_j (\partial \mathbf{r}_i / \partial q_j)\delta q_j = \mathbf{J}_i \delta \mathbf{q} \neq \mathbf{0} \tag{26}$$

the (8) can be stated as:

$$\delta \mathbf{p}_i = \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q})(\mathbf{J}_i^k(\mathbf{q})\delta \mathbf{q})$$
$$+ (\mathbf{R}_i'(\mathbf{q})\delta \mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q})\mathbf{u}_i^k$$
$$+ \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)(\mathbf{T}_k^{i'}(\mathbf{q})\delta \mathbf{q})\mathbf{u}_i^k \tag{27}$$

The second and third terms count the $\mathbf{u}$ vector that is small, since we are in the *small displacement condition*. Thus, they can be neglected being their order of magnitude lower with respect to the first term in (27) that becomes:

$$\delta \mathbf{p}_i = \mathbf{R}_i(\mathbf{q})\mathbf{N}_i(x_i, y_i, z_i)\mathbf{T}_k^i(\mathbf{q})\mathbf{J}_i^k(\mathbf{q})\delta \mathbf{q} \tag{28}$$

By substituting (9), (13) and (28) into the (4), the following dynamic equation holds:

$$\sum_i \int_{v_i} [\delta \mathbf{q}^T \mathbf{J}_i^{kT} \mathbf{T}_{k,i}^{i\,T} \mathbf{N}_i^T \mathbf{R}_i^T][\mathbf{R}_i \mathbf{N}_i \mathbf{T}_{k,i}^i \ddot{\mathbf{r}}_i^k + \mathbf{R}_i \mathbf{N}_i \mathbf{T}_{k,i}^i \ddot{\mathbf{u}}_i^k$$
$$+ 2(\dot{\mathbf{R}}_i \mathbf{N}_i \mathbf{T}_{k,i}^i + \mathbf{R}_i \mathbf{N}_i \dot{\mathbf{T}}_{k,i}^i)\dot{\mathbf{u}}_i$$
$$+ (\ddot{\mathbf{R}}_i \mathbf{N}_i \mathbf{T}_k^i + 2\dot{\mathbf{R}}_i \mathbf{N}_i \dot{\mathbf{T}}_k^i + \mathbf{R}_i \mathbf{N}_i \ddot{\mathbf{T}}_k^i)\mathbf{u}_i^k]\rho_i dv$$
$$+ \sum_i \int_{v_i} (\mathbf{u}_k^{i\,T} \delta \mathbf{T}_{k,i}^{iT} \mathbf{B}_i^T)\mathbf{D}_i \mathbf{B}_i \mathbf{T}_{k,i}^i \mathbf{u}_i^k dv$$
$$= \sum_i \int_{v_i} (\delta \mathbf{q}^T \mathbf{J}_i^{kT} \mathbf{T}_{k,i}^{iT} \mathbf{N}_i^T \mathbf{R}_i^T)\mathbf{g}\rho_i dv + \delta \mathbf{r}^T \mathbf{f} \tag{29}$$

The $\delta \mathbf{T}_{k,i}^{iT}$ terms in the elastic virtual work term can be transformed into an equivalent form by means of the virtual displacement of the generalized coordinates as stated in (24-26). Moreover the integrals that rise from the inertia virtual work term are the same previously evaluated for the local nodal equilibrium in (15-22). Thus, finally it holds the following for the global equilibrium:

$$\delta \mathbf{q}^T \mathbf{J}^T [\mathbf{M}(\ddot{\mathbf{r}} + \ddot{\mathbf{u}}) + 2(\mathbf{M}_{G1} + \mathbf{M}_{G2})\dot{\mathbf{u}} + (\mathbf{M}_{C1} + 2\mathbf{M}_{C2} + \mathbf{M}_{C3})\mathbf{u}]$$
$$+ \sum_i \mathbf{u}_i^T \delta \mathbf{q}^T \mathbf{K}_{1,i} \mathbf{u}_i = \delta \mathbf{q}^T \mathbf{J}^T (\mathbf{f}_g + \mathbf{f}) \tag{30}$$

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The integration scheme for the equations of motion With the local nodal and global equilibrium, two different and independent set of equations have been found. Substituting the local equilibrium, (23), into the global equilibrium, (30), the following holds:

$$\sum_i \mathbf{u}_i^T \delta\mathbf{q}^T \mathbf{K}_{1,i}\mathbf{u}_i - \sum_i \delta\mathbf{q}^T \mathbf{J}_i^T \mathbf{K}_i \mathbf{u}_i = 0 \tag{31}$$

For the small displacement assumption, the first term can be neglected, since $\mathbf{u}^T\delta\mathbf{q}^T\mathbf{K}_{1,i}\mathbf{u}$ is negligible with respect to $\delta\mathbf{q}^T\mathbf{J}_i^T\mathbf{K}_i\mathbf{u}_i$. Hence, by computing the sums for the whole elements of the whole mechanism under analysis, the following differential system, encompassing both the local nodal and global equilibrium holds

$$\begin{aligned}&\delta\mathbf{u}^T\left[\mathbf{M}(\ddot{\mathbf{r}}+\ddot{\mathbf{u}})+2(\mathbf{M}_{G1}+\mathbf{M}_{G2})\dot{\mathbf{u}}+(\mathbf{M}_{C1}+2\mathbf{M}_{C2}+\mathbf{M}_{C3})\mathbf{u}\right]\\ &+\ \delta\mathbf{u}^T\mathbf{K}\mathbf{u}=\delta\mathbf{u}^T(\mathbf{f}_g+\mathbf{f})\end{aligned} \tag{32}$$

$$\begin{aligned}&\delta\mathbf{q}^T\mathbf{J}^T\left[\mathbf{M}(\ddot{\mathbf{r}}+\ddot{\mathbf{u}})+2(\mathbf{M}_{G1}+\mathbf{M}_{G2})\dot{\mathbf{u}}+(\mathbf{M}_{C1}+2\mathbf{M}_{C2}+\mathbf{M}_{C3})\mathbf{u}\right]\\ &=\delta\mathbf{q}^T\mathbf{J}^T(\mathbf{f}_g+\mathbf{f})\end{aligned} \tag{33}$$

The infinitesimal displacements of the ERLS can be expressed by means of the Jacobian matrix so that $\delta\mathbf{u}$'s, the $\delta\mathbf{q}$'s can be eliminated from the last two equations; moreover, to fit better this system, friction is taken into account by using the Rayleigh model of damping yielding:

$$\begin{aligned}&\mathbf{M}(\ddot{\mathbf{r}}+\ddot{\mathbf{u}})+2(\mathbf{M}_{G1}+\mathbf{M}_{G2})\dot{\mathbf{u}}+\alpha\mathbf{M}\dot{\mathbf{u}}+\beta\mathbf{K}\dot{\mathbf{u}}\\ &+\ (\mathbf{M}_{C1}+2\mathbf{M}_{C2}+\mathbf{M}_{C3})\mathbf{u}+\mathbf{K}\mathbf{u}=\mathbf{f}_g+\mathbf{f}\end{aligned} \tag{34}$$

$$\begin{aligned}&\mathbf{J}^T\mathbf{M}(\ddot{\mathbf{r}}+\ddot{\mathbf{u}})+2\mathbf{J}^T(\mathbf{M}_{G1}+\mathbf{M}_{G2})\dot{\mathbf{u}}+\alpha\mathbf{J}^T\mathbf{M}\mathbf{u}\\ &+\ \mathbf{J}^T(\mathbf{M}_{C1}+2\mathbf{M}_{C2}+\mathbf{M}_{C3})\mathbf{u}=\mathbf{J}^T(\mathbf{f}_g+\mathbf{f})\end{aligned} \tag{35}$$

where, in the (34) the nodal equilibrium is posed, i.e. equivalent loads applied to each node must be in equilibrium, whilst in the (35) the global equilibrium is posed, i.e. all equivalent nodal loads applied to the linkage produce no work for a virtual displacement of the ERLS.

These equations can be grouped and rearranged in matrix in order to put in evidence accelerations that will be used in a double integration scheme in order to get position and displacement of the robots

$$\begin{aligned}&\begin{bmatrix}\mathbf{M} & \mathbf{M}\mathbf{J}\\ \mathbf{J}^T\mathbf{M} & \mathbf{J}^T\mathbf{M}\mathbf{J}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{u}}\\ \ddot{\mathbf{q}}\end{bmatrix}=\\ &\begin{bmatrix}-2(\mathbf{M}_{G1}+\mathbf{M}_{G2})-\alpha\mathbf{M}-\beta\mathbf{K}-\mathbf{M}\dot{\mathbf{J}}-(\mathbf{M}_{C1}+2\mathbf{M}_{C2}+\mathbf{M}_{C3})-\mathbf{K}\\ \mathbf{J}^T(-2(\mathbf{M}_{g1}+\mathbf{M}_{g2})-\alpha\mathbf{M})-\mathbf{J}^T\mathbf{M}\mathbf{J}-\mathbf{J}^T(\mathbf{M}_{C1}+2\mathbf{M}_{C2}+\mathbf{M}_{C3})\end{bmatrix}\begin{bmatrix}\dot{\mathbf{u}}\\ \dot{\mathbf{q}}\\ \mathbf{q}\end{bmatrix}+\\ &+\begin{bmatrix}\mathbf{M} & \mathbf{I}\\ \mathbf{J}^T\mathbf{M} & \mathbf{J}^T\end{bmatrix}\begin{bmatrix}\mathbf{g}\\ \mathbf{f}\end{bmatrix}\end{aligned} \tag{36}$$

Has to be noted that the coefficient matrix is not square, since the generalized coordinate $\mathbf{q}$ has been added in the column of the unknowns vector. As a consequence, for a given configuration d**b** of infinitesimal nodal displacements corresponds to more sets

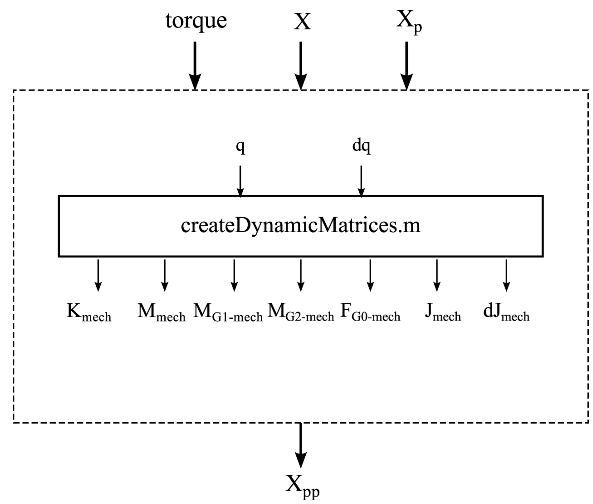

FIGURE 3. Flow-chart of the *dynamic.m* Matlab function that is the core of the integration scheme for the simulink model; $X_{pp} = \ddot{X} = [\ddot{u};\ddot{q}]$, $X_p = \dot{X} = [\dot{u};\dot{q}]$, $X = [u;q]$

of increments $\begin{bmatrix} d\mathbf{u}^T & d\mathbf{q}^T\end{bmatrix}$ of the generalized coordinates of the system. The way used in the follow to eliminate this redundancy will be to force to zero a number of elements of d**u** equal to the number of generalized coordinates of the ERLS.

A Matlab implementation for the ERLS 3D model

In the previous section a mathematical structure based on the ERLS 3D-model was presented. In order to use it, a software implementation of this model has to be stated. In previous work [12], matrix computing and construction were arranged in a serial way, e.g. the mechanism was assembled adding sequentially the links starting from the base one and arriving to end-effector one. For general parallel manipulators is not further possible to use this system, since more than one closed loop-chains can be stated in the kinematic model.

Moreover, considering $Matlab^{TM}$ as the adopted software, a serial or single-closed chain structure with a reduced number of links can be treated by means of a fully symbolic approach. For general parallel manipulators this is not any more suitable, since the number of links grows and there is not any way to automate the kinematics analysis as for the serial manipulators and planar systems. Hence, a different $Matlab^{TM}$ implementation is here proposed by combining symbolic and numeric approaches to allow the numeric kinematic analysis and, in the meanwhile, get the faster simulation with the higher versatility.

The integration scheme has been implemented by means of the $Simulink - Matlab^{TM}$ toolbox and the core of the system is represented by the so-called *dynamic.m* function which structure is presented in Fig.3.

The *createDynamicMatrices.m* function that appears in

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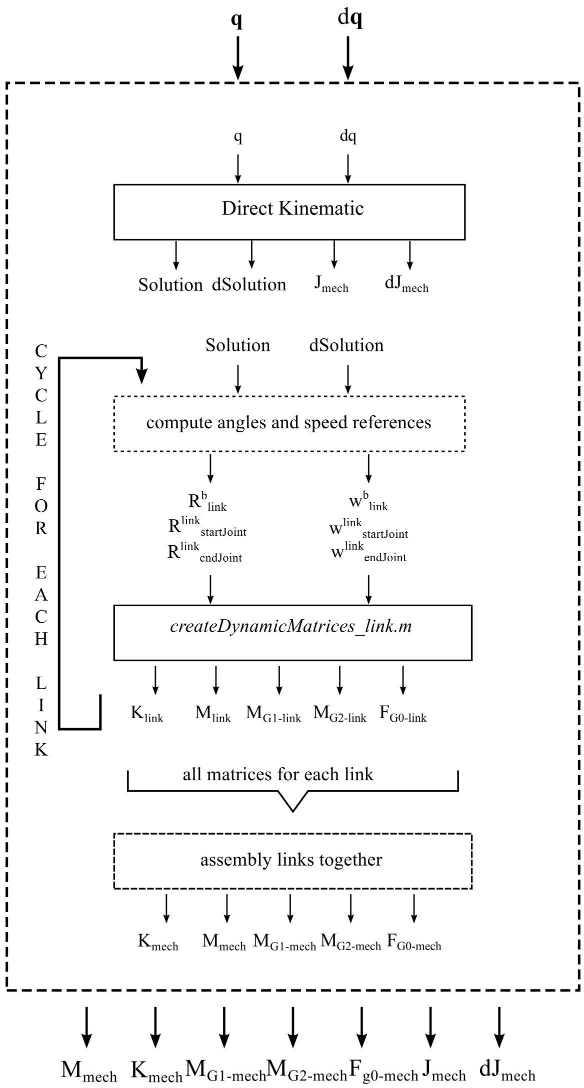

FIGURE 4. Flow-chart of the *createDynamicMatrices.m* function that is used online in order to create the needed dynamic matrices for the ERLS model

Fig.3 is shown in Fig.4 in a proper form: it takes as input the vector of position $\mathbf{q}$ and speed $\dot{\mathbf{q}}$ of the free coordinates; as output, it returns the mass, damping, Coriolis, gravity matrices $\mathbf{M}_{mech}$, $\mathbf{K}_{mech}$, $\mathbf{M}_{g1-mech}$, $\mathbf{M}_{g2-mech}$, $\mathbf{M}_{c-mech}$ and $\mathbf{F}_{mech}$. For serial robots, this function can represent only a substitution: the matrices are off-line computed in a symbolic way as a function of the free coordinates of the system. Since the direct kinematics of parallel manipulators is pretty complicate and it is not more feasible to manage such a complicate symbolic structure or it has not an analytical description, this approach is not more possible. A trade-off between numerical and symbolic approach has been taken, since a fully numeric approach can result very time-consuming. For these issues, the function under consideration has been completely rearranged. Inside it, four parts can be highlighted:

-Kinematics analysis
-Compute useful rotation matrices and speed vectors
-Compute dynamic *link* matrices
-Assembly dynamic *mechanism* matrices

The symbolic off-line computing is limited to the dynamic link matrices. So, in the on-line procedure, it is enough to compute the generalized coordinates position and speed, and substitute them into the off-line computed dynamic link matrices.

Simulation and validation

The software has been firstly tested for some serial robots in order to be sure that the results of the new software are in agreement with the previous formulation, so that the extension for parallel robots is still able to take into account serial robots. After that, in order to apply the ERLS model for parallel manipulators, different robots such as a 3-RRR robot, a PacDrive robot [14] and a Delta robot [15] have been implemented. In this work, the PacDrive robot results are presented. The PacDrive [14] robot is a high dynamic performance pick and place robot. It's a lower cost opportunity than the full Delta robot, since its kinematics is planar since the end-effector can only move in the *xy* plane as shown in fig.5.

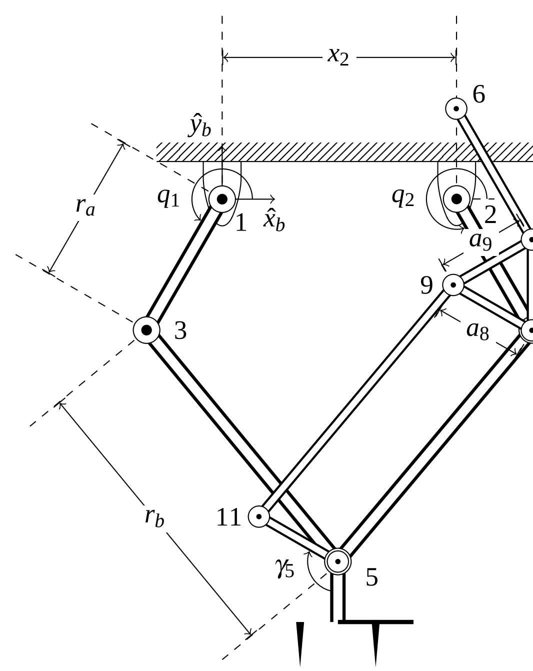

FIGURE 5. The PacDrive with relevant lengths and joints numbers

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link	length	ρ	E	b_y	c_z	ν
	[m]	$[Kg/m^3]$	$[N/m^2]$	[m]	[m]	[/]
1	0.20	7840	$5 \cdot 10^9$	0.02	0.03	0.33
2	0.20	7840	$5 \cdot 10^9$	0.02	0.03	0.33
3	0.50	7840	$5 \cdot 10^9$	0.02	0.03	0.33
4	0.50	7840	$5 \cdot 10^9$	0.02	0.03	0.33
5	0.05	7840	$5 \cdot 10^9$	0.01	0.03	0.33
6	0.20	7840	$5 \cdot 10^9$	0.01	0.02	0.33
7	0.07	7840	$5 \cdot 10^9$	0.01	0.02	0.33
8	0.07	7840	$5 \cdot 10^9$	0.01	0.02	0.33
9	0.07	7840	$5 \cdot 10^9$	0.01	0.02	0.33
10	0.50	7840	$5 \cdot 10^9$	0.01	0.02	0.33
11	0.07	7840	$5 \cdot 10^9$	0.01	0.02	0.33

TABLE 1. The chosen mechanical parameters for the PacDrive robot: E is the flexibility modulus, b_y and c_z the size of the link respectively in the y and z axis, ν is the Poisson's ratio.

The structure can be viewed as a couple of two frames: the main frame is the Pentagon connecting joints 1, 3, 5, 4, 2, that has two rigid degrees of freedom, namely q_1 and q_2. The other part is a passive frame, useful to get the end-effector always parallel to the ground (like in the full Delta robot). This can be done with a parallelogram chain (joints 6,7,4,2) that keeps the 4-7 link always parallel with the 6-2 one (that is fixed); then a triangle (4,7,9) that keeps the 9-4 link in a fixed angular relation with the 4-7 link; finally a further parallelogram (4,9,11,5) keeps the 5-11 link parallel with the 9-4 one. With this choice, if the joints 2 and 6 lies on the same x coordinate of the base-frame and the triangle (4,7,9) is equilateral, the 5-11 link always result at 150 degree in the base frame. If $\gamma_5 = 120$ degrees, the end-effector will be always perpendicular with the ground. Notice that this structure, added in order to get the end-effector always in a known rotation angle, is fully passive, so it does not need any actuated joint.

The parallel mechanism simulated with the ERLS model has been compared with $Adams^{TM}$, one of the most widely used software for multibody simulation. The chosen mechanical parameters, similar to those values used in effective existing robots, are itemized in tab.1.

In tab.2, for each link the starting and ending joints are itemized, in order to correctly apply the rules for the enumerating the degrees of freedom. In the same table, it is possible to see the number of beams that have been chosen for modeling each link.

link	starting joint	ending joint	beams
1	1	3	2
2	2	4	2
3	3	5	2
4	4	5	2
5	5	0	1
6	6	7	2
7	7	4	1
8	4	9	1
9	9	7	1
10	9	11	2
11	11	5	1

TABLE 2. The starting and ending joints for each link, along with the choice for the number of beams, i.e. the number of Euler-Bernoulli finite elements in which the considered link is subdivided

Rigid link simulation The first test carried out show a PacDrive with very rigid links, in order to validate the kinematics of the model. The flexibility parameter has been fixed to $E = 5 \cdot 10^{12} N/m^2$. In this first analysis the PacDrive robot is left from the initial position $q_1 = -170^\circ$ and $q_2 = -10^\circ$ under gravity. At $t = 0.1sec$ a torque of $20Nm$ is applied to both the actuated links in order to rise them and contrast the force of gravity. At $t = 0.2sec$ the external torque is removed and the only force of gravity is applied to the system. There are no initial speeds.

In fig.6 the free coordinates of the PacDrive robot are shown: as it can easily observed, the curves are in optimal agreement with the curves obtained with the $Adams^{TM}$ simulator.

Flexible link simulation For a better validation of the ERLS-based PacDrive robot, it has been chosen to test it with quite flexible links, i.e. with a flexibility parameter equals to $E = 5 \cdot 10^9 N/m^2$. As before, the PacDrive robot is left from the initial position $q_1 = -170^\circ$ and $q_2 = -10^\circ$ to the only force of gravity. At $t = 0.1sec$ a torque of $20Nm$ is applied to both actuated links in order to rise them as opposite as the work of the force of gravity. At $t = 0.2sec$ the external torque is removed and the only force of gravity is applied to the system. There are no initial speeds.

In fig.7 the free coordinates of the PacDrive are shown: as it can easily observed, the curves obtained with our ERLS simulator overlap almost perfectly with those obtained with $Adams^{TM}$; due to the closed-chain kinematics of the system the gross motion of the system results very similar to the rigid link case.

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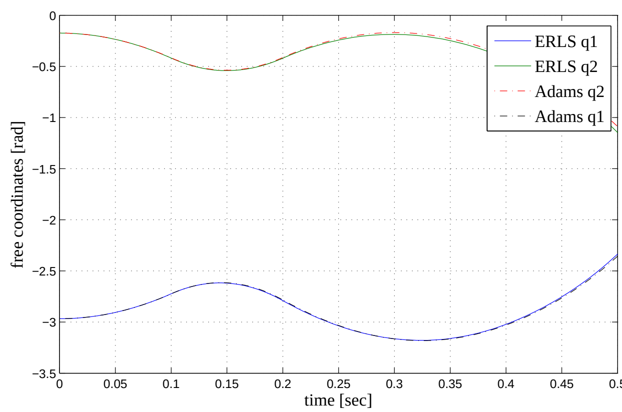

FIGURE 6. Free coordinates q_1 and q_2 of the PacDrive robot in the rigid link simulation

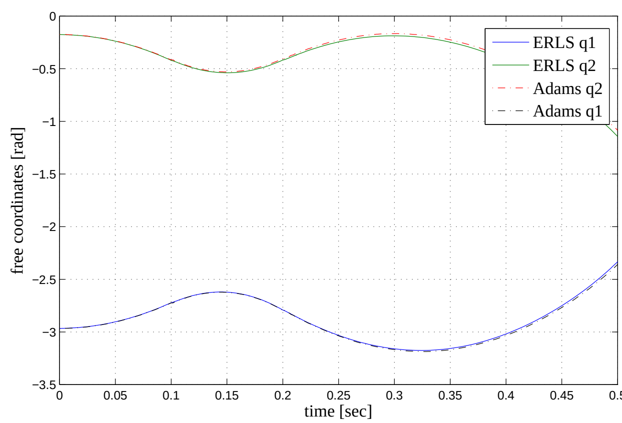

FIGURE 7. Free coordinates q_1 and q_2 of the PacDrive in the $flexible$ link simulation

However, in fig.8 the x displacement of the tip of the PacDrive is shown. In this case it can be observed that there is a significant deformation, due to the more flexible links. The task of the designer of the controller will be compensate this vibration/deformation in order to make the manipulator the desired trajectory.

After these tests it is possible to conclude that the new ERLS-based model for parallel manipulators is well fitting with the old model, so this extension is still able to work with the serial manipulators, and that the new model is even well working with parallel manipulators.

A general pick-and-place trajectory As a final simulation, it has been chosen to test the PacDrive robot under a general pick-and-place stress. Over the force of gravity, a periodic square torque with period $T = 0.2sec$ and duty cycle equal to $\delta = 0.5$ has been applied. The entity of this torque is $15Nm$ when it's against the force of gravity, and $5Nm$ in the other case. The PacDrive robot is left from the initial position $q_1 = -140°$ and $q_2 = -40°$ and the initial speed of q_1 is $3rad/sec$, the Opposite for the q_2 free coordinate. The other mechanical parameters are the same of those adopted in tab.1. With these parameters the Pac Drive will make an up-and-down trajectory, where the links will be quite stressed. In fig.9 the free coordinates and their periodic behaviour are shown. In fig.10 and 11 the x and y displacement of the tip of the PacDrive are shown. Since it is made with flexible-links, the deformations are not negligible.

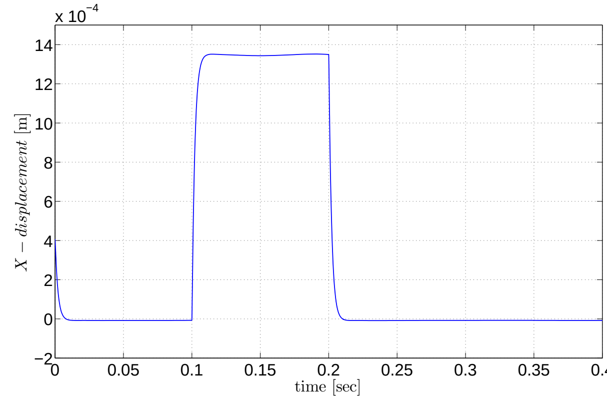

FIGURE 8. Displacement of the x coordinate of the PacDrive's tip in the $flexible$ links simulation

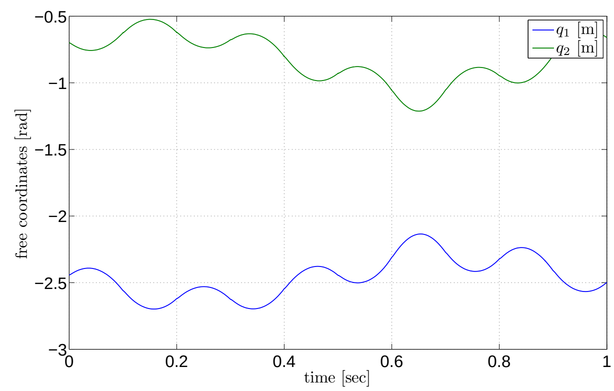

FIGURE 9. Free coordinate q_1 and q_2 of the PacDrive in the pick-and-place trajectory simulation

Conclusions and future work

In this work, the issue of accurately modelling the dynamics of lightweight flexible-link parallel robots has been investigated by means of an Equivalent Rigid-Link System (ERLS) formula-

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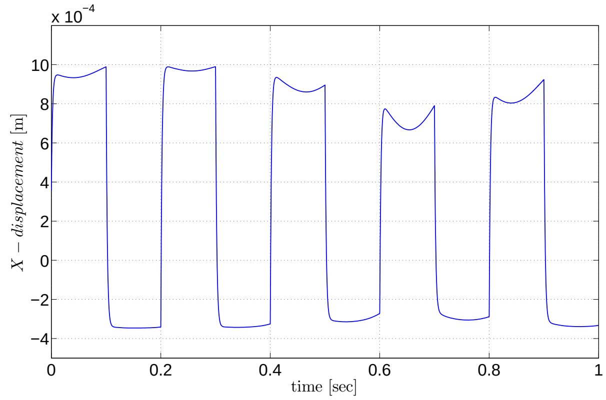

FIGURE 10. Displacement of the *x* coordinate of the PacDrive's tip in the pick-and-place trajectory simulation

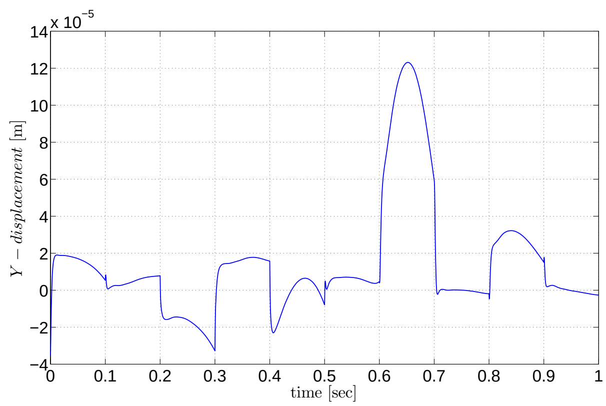

FIGURE 11. Displacement of the *y* coordinate of the PacDrive's tip in the pick-and-place trajectory simulation

tion. This formulation, useful for describing the dynamic evolution of 3D serial robots with flexible-links, has been extended to Parallel- Kinematic-Machines (PKM) both from the theoretical and from the software implementation points of view. A simulator, capable of reproducing the dynamics of the system and predicting the deformations due to the elasticity of the links, has been developed. The PacDrive robot model has been considered as test-case in order to validate the system. Futur work will cover the experimental validation of the model and the use of the formulation in order to set-up effective controllers capable to predict and avoid or limit vibrations on flexible-link parallel kinematic machines.

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