A model-based trajectory planning approach for flexible-link mechanisms

Paolo Boscariol*, Alessandro Gasparetto*, Renato Vidoni†, Armando Romano*
*Dipartimento di Ingegneria Elettrica, Gestionale e Meccanica, Università di Udine
Via delle Scienze 208, 33100 Udine, Italy
paolo.boscariol@uniud.it
†Faculty of Science and Technology, Free University of Bolzano-Bozen
Piazza Università 5, 39100 Bolzano, Italy 33100 Udine, Italy

***Abstract*—In this paper the problem of trajectory planning for flexible-links mechanisms is dealt with. The method proposed here is suitable for the determination of model-based optimal point-to-point trajectories with bounds on kinematic and dynamic characteristics of the mechanism. An open-loop optimal control strategy is applied to an accurate dynamic model of flexible multi-body planar mechanisms. The model, which has already benn fully validated through experimental tests, is based on finite element discretization and accounts for the main geometric and inertial nolinearities of the linkage. Exploiting an indirect or variational solution method, the necessary optimality conditions deriving from the Pontryagin's minimum principle are imposed, and lead to a differential Two-Point Boundary Value Problem (TPBVP); numerical solution of the latter is accomplished by means of collocation techniques. Considering a lightweight RR robot, simulation results are provided for rest-to-rest trajectories with bounded speed and bounded elastic deformation. However, the strategy under investigation has general validity and can be applied to other types of machanisms, as well as with different objective functions and boundary conditions.**

I. Introduction

High speed operation is a recurring target in design and application of robotic manipulators, for clear economic reasons. Also maximizing the ratio between the weight of the payload and that of the whole mechanism is a common objective. Traditionally, to ensure a good position accuracy of manipulators, their arms have been designed and built to behave like rigid bodies. Hence, conventional robots still present heavy links and bulky structures. However, such systems are shown to be inefficient in terms of actuator power consumption and speed, related to their load-carrying capacity [1], as payload-to-manipulator weight ratio is typically ranging from 1:100 to 1:10 [2].

On the other hand, a common trend of robot design is to develop lightweight mechanisms also for industrial applications, not only for weight-critical tasks such as outer space explorations. Lighter robots exhibit several advantages over heavy rigid ones, such as lower cost, improved energy efficiency and safety, higher operating speed and payload-to arm weight ratio. Despite this, lower arm stiffness generates flexibility and vibration issues. In these cases, dynamic analysis and control strategies based on the rigid-link assumption turn out to be no longer adequate: structural flexibility, if neglected or poorly controlled, can lead to major worsening in the accuracy of positioning and motion, to high mechanical stress, and also to instability.

During the last 30 years, large efforts have been made in both academic and industrial settings to offer solutions to the aforementioned problems. Several approaches have been explored, focusing on the main tasks of dynamic modeling, control and trajectory planning of Flexible-Link Manipulators (FLM) [1]. While a large number of model-independent algorithms are available in literature [3], some of which have been tested also on mechanisms both joint [4] and link [5] flexibility, less attention has been cast on model-based approaches.

FLMs are continuous nonlinear dynamical systems, possessing an infinite number of elastic degrees of freedom. A crucial aspect is thus represented by the strategy adopted to obtain a finite-dimensional approximation of their dynamics. The main existing approaches are Assumed Mode Method (AMM) and Finite Element Method (FEM), commonly used in combination with the Lagrangian or the recursive Newton-Euler formulations for deriving the system equations of motion. Lumped Parameter modeling, i.e. approximation by means of spring-mass systems, is rarely chosen [6] due to its limited accuracy. If linearized models around a specified operating point are taken into consideration, as done in [7], the dynamics of multi-link FLM is described with limited accuracy: as proved experimentally by Milford and Asokanthan in [8], the eigenfrequencies of a two-link flexible robot can vary up to 30% as a function of the manipulator configuration. Moreover, numerical and experimental studies [9], [10] have demonstrated that an accurate dynamic modeling of FLMs must consider both the coupling between rigid-body and elastic motions and the main geometric and inertial nonlinearities. In Assumed-Mode Method, only a set of eigenfrequencies are used to describe the flexible behavior of the manipulators, along their whole operative range [6].FEM formulation is reputed to be more accurate than the AMM in describing flexible multi-link manipulator dynamics, and arms with complex cross-sectional geometries too, as reported by Theodore and Ghosal in [6]. In addition, Lee showed in [11] that conventional Lagrangian modeling of FLMs is not very accurate, in case of rotational motion of the links.

This work concerns the development of point-to-point optimal path planning algorithms, for planar mechanisms with flexible arms. An highly accurate nonlinear dynamic model of FLMs, based on finite element discretization and Equivalent Rigid-Link System (ERLS) formulation, is used. Global differential equations of motion are obtained by direct application of the principle of virtual work, and they account for the mutual inertial influence between elastic and rigid-body motion [12].

In point-to-point trajectory optimization problems, only the initial and final end-effector positions are given, and the manipulator is free to move between them. The path is therefore subject to optimization, and it is selected with the aim of minimizing a cost functional.

The topic of model-based optimal trajectory planning of flexible-link mechanisms is somehow limited, while a large number of strategies has been developed for rigid manipulators [13].The feedforward techniques used in this paper do not require any additional sensors. They are thus more economical then closed-loop strategies, for vibration control of robotic manipulators performing repetitive tasks [14], [15].

In direct methods, the original optimal control problem is converted into a parameter optimization one [16], by discretizing robot dynamic variables (states and/or controls). Then an efficient deterministic or stochastic optimization algorithm can be applied to solve this new finite-dimensional problem. In [17] and [18], residual vibration reduction is attained approximating the joint motion profiles with splines or polynomial functions.A similar approach is proposed in [19], in which optimal rest-to-rest motion for a two-flexible-link robot is evaluated, using genetic algoritms and polynomial functions. As in most direct approaches to model-based trajectory optimization, dynamic modeling is obtained through the Assumed-Mode Method. An exception is represented by [20]: Finite Element Method is applied to a flexible RR manipulator, in combination with a Genetic Algorithm–fuzzy logic feedback control strategy. Also in [2], [21], a FEM-based modeling for a two-link flexible manipulator is employed: but in this case, an open-loop discrete dynamic programming (DDP) path-planning scheme is proposed. Moreover, as a result of the control parametrization introduced, all direct methods can only yield approximate solutions to the optimal control problem. Due to the large number of parameters involved, they are extremely time-consuming and quite inefficient, especially for systems with a large number of elastic d.o.f. [22].

Indirect methods make use of calculus of variation: necessary conditions for optimality deriving from the Pontryagin's Minimum Principle (PMP) are imposed, and the resulting Two-Point Boundary Value Problem (TP-BVP) is solved, by suitable numerical techniques. Indirect methods are widely reckoned to be very accurate, particularly when a large number of elastic d.o.f. is present, or optimization of composite objectives is targeted [23]. In [22], Korayem et al. have developed an algorithm for the point-to-point motion planning of a two-link FLM with revolute joints. Euler-Lagrange formulation and Assumed-Mode Method are used to describe the dynamics of the robot. The cases of minimum effort, minimum effort-speed, maximum payload and minimum vibration are examined; only constraints on joint actuator torques are imposed.

To the best of the authors' knowledge, the only paper applying both FEM-based dynamic modeling and indirect solution methods to the open-loop optimal control problem of FLMs is [24], again by Korayem et al. In it, however, Lagrangian formulation is used to define the robot dynamics, and results are provided only for time evolution of joint speeds and motor torques. Therefore, no results in terms of vibration are presented. Moreover, only minimum effort-speed trajectories are investigated.

In this paper an indirect strategy is developed for the optimal path planning of flexible multi-link manipulators. Constraints on kinematic and dynamics characteristics of the motion of the mechanisms are introduced as well. It will be shown in the paper that the use of special penalty function allows to impose constraints on the speed of the joint of the robot, and on some characteristics of the response of the mechanisms, such as on the amplitude of elastic deformation.

II. Dynamic modelling

In this section a brief explanation of the dynamic model used for the definition of the trajectory planning problem is given. Such formulation, introduced by Giovagnoni in [12] and extended in [25], [26], is based on FEM discretization and on the principle of virtual works. The resulting model accounts for the inertial non-linearities of the mechanisms and gives a coupled description of both the rigid and flexible motion of a planar FLM with an arbitrary number of links. It has been validated in several works, including [27], [28]. Owing to the space constraints of this paper, only a brief overview of the model will be included. Fore a more detailed discussion on the subject, see [12], [29].

Each flexible link belonging to the mechanism is subdivided into finite elements. The motion of the mechanism can be thought as the superposition of the motion of an equivalent rigid-link system (ERLS) and the elastic motion of the nodes of the finite elements. Therefore the free coordinates of the system are the angular position of the two links (vector $\mathbf{q}$) and the vector of the nodal displacements $\mathbf{u}$. The dynamics of a generic planar flexible links mechanism is described by:

$$\begin{bmatrix} \mathbf{M} & \mathbf{MS} & \mathbf{0} & \mathbf{0} \\ \mathbf{S}^T\mathbf{M} & \mathbf{S}^T\mathbf{MS} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{q}} \\ \dot{\mathbf{u}} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \mathbf{I} \\ \mathbf{S}^T\mathbf{M} & \mathbf{S}^T \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \tau \end{bmatrix}$$

$$+ \begin{bmatrix} -2\mathbf{M}_G - \alpha\mathbf{M} - \beta\mathbf{K} & -\mathbf{M}\dot{\mathbf{S}} & -\mathbf{K} & \mathbf{0} \\ \mathbf{S}^T(-2\mathbf{M}_G - \alpha\mathbf{M}) & -\mathbf{S}^\mathbf{T}\mathbf{M}\dot{\mathbf{S}} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{q}} \\ \mathbf{u} \\ \mathbf{q} \end{bmatrix} \tag{1}$$

in which $\mathbf{g}$ and $\mathbf{F}$ are the vector of gravitational forces and the vector of generalized external forces acting, respectively. It should be pointed out that in this model the mentioned rigid and elastic motions are totally coupled each other. The nomenclature used in (1) is reported in Table 1.

TABLE I
NOMENCLATURE USED IN EQUATION (1)

symbol	description
$\mathbf{q}$	vector of the free coordinates
$\mathbf{u}$	vector of nodal displacements
$\mathbf{M}$	mass matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{S}$	sensitivity coefficient matrix
$\mathbf{I}$	identity matrix
$\mathbf{M}_G$	matrix of Coriolis acceleration contributions
$\mathbf{g}$	vector of gravity forces
τ	vector of generalized external forces
α, β	Rayleigh damping constants

In order to define correctly the minimization problem that is used to evaluate an optimal motion profile, the direct dynamics of the flexible-link manipulator under consideration must be computed in its symbolic form. The reason of this need will be explained in the following section. A more compact form of the system in eq. 1 is:

$$\tilde{\mathbf{M}}\dot{\mathbf{x}} = \Phi(\mathbf{x}, \mathbf{F}, \mathbf{t}) \tag{2}$$

The direct dynamics of the manipulator is the first order ODE system:

$$\dot{\mathbf{x}} = \tilde{\mathbf{M}}(\mathbf{x}, t)^{-1}\Phi(\mathbf{x}, \mathbf{F}, t) \tag{3}$$

which must be computed in its symbolic explicit form. In order to do this, a symbolic algebraic tool must be used. In this case all the computation has been done using Matlab Symbolic Toolbox, with the same procedure presented in [30]. The mechanism under consideration in this paper is not affected by gravity, therefore the gravitational term $\mathbf{g}$ in equation (1) is null. Moreover, if only rotary actuators are used, the vector of generalized forces $\mathbf{F}$ includes only torques and null terms. If τ_i is the torque produced by the i-th actuator and τ is the vector containing all the τ_i, $\mathbf{F}$ is simply $\mathbf{F} = \mathbf{F}(\tau)$. Therefore equation (3) can be rewritten as:

$$\dot{\mathbf{x}} = \tilde{\mathbf{M}}(\mathbf{x}, t)^{-1}\Phi(\mathbf{x}, \tau, t) = \Omega(\mathbf{x}, \tau, t) \tag{4}$$

III. FORMULATION OF CONSTRAINED MOTION PLANNING PROBLEM

The target of this study is to find a way to compute a trajectory that brings the plant form a given initial condition $\mathbf{x}(t_0) = \mathbf{b}_0$ to the final configuration $\mathbf{x}(t_f) = \mathbf{b}_f$ in a given time. Among the infinite number of choices, the motion profile that we are looking for is the solution of the following optimization problem:

$$\begin{cases} \min\limits_{\tau} J = \int\limits_{t_0}^{t_f} f(\mathbf{x}(t), \tau, t)dt + Z(\mathbf{x}(t), \tau, t_f) \\ subject\ to: \\ \dot{\mathbf{x}}(t) = \Omega(\mathbf{x}(t), t, \tau) \\ \mathbf{X}^- \leq \mathbf{x} \leq \mathbf{X}^+ \\ \mathbf{x}(t_0) = \mathbf{b}_0 \\ \mathbf{x}(t_f) = \mathbf{b}_f \end{cases} \tag{5}$$

f is a smooth differential function of the state variable $\mathbf{x}$ and of the control vector τ, while Z is a function of the system at terminal time, therefore is often referred as the terminal cost. Since the problem is constrained by the nonlinear dynamic system $\Omega(\mathbf{x}(t), t, \tau)$ which represents the dynamics of the manipulator, the problem above is a nonlinear constrained optimization problem. One of the ways to solve it is to formulate a TPBVP (Two-Point Boundary Value Problem) trough the use of Pontryagin Minimum Principle (PMP) and Hamilton-Jacobi-Bellman (HJB) equation.

The first step is to define the Hamiltonian, i.e. a function in the form:

$$\mathcal{H}(\mathbf{x}(t), t, \tau) = J + \Lambda^T\Omega(\mathbf{x}(t), \tau, t) \tag{6}$$

where Λ is the vector of Lagrangian multipliers:

$$\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]^T$$

whose length is n, which equals the size of the state of the plant: $\mathbf{x}(t) \in \mathbf{R}^n$. Λ is also often called costate vector . Pontryagin Minimum Principle states that the necessary conditions to obtain a solution to the problem stated in equation (5) are:

$$\dot{\mathbf{x}} = \frac{\partial \mathcal{H}}{\partial \Lambda}; \quad \dot{\Lambda} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}; \quad 0 = \frac{\partial \mathcal{H}}{\partial \tau} \tag{7}$$

Among the infinite state trajectory that satisfy the three conditions stated above, our goal is to choose the ones that respect the two boundary conditions at initial and final time t_0 and t_f. In general, such conditions can be posed on the so called augmented state, i.e. $\mathbf{y}(t) = [\mathbf{x}(t), \Lambda(t)]$, but in most cased such conditions are posed on the sole state vector of the plant $\mathbf{x}(t)$, since they have a straightforward physical meaning.

A. Constrained solution

The optimization problem stated in eq. (5) includes some constraints on the state vector of the plant $\mathbf{x}$, which must be respected for all the duration of the trajectory. Hard constraints however cannot be included in the cost function J that appears in (5), since using a function as the step function would make the function J non differentiable, which is incompatible with the second PMP condition (7). A possible solution to this is to include an approximation of the step function into the functional J. As proposed in [31], the following function can be used:

$$p(w, \sigma) = w + \frac{1}{\sigma}log(1 + e^{-\sigma w}) \tag{8}$$

which is a C^∞ approximation of the function $w_+ := max, 0\}$. w is the value to be constrained, while σ is a parameter which controls the "sharpness" of the function p at $w = 0$. Therefore the higher the value of the parameter σ, the better approximation of the step function is obtained. On the other hand, higher values of σ might make the numerical solution of the problem (5) ill-conditioned. A good technique to solve this problem is to use the continuation method

	symbol	value
Young's modulus	E	70×10^9 Pa
Flexural inertia moment	J	8.333×10^{-10} m^4
Beam width and thickness	a	10 mm
Length of first link	L_1	0.5 m
Length of second link	L_2	0.5 m
Mass/unit of length of links	m	0.27 kg/m
Concentrated mass at second joint	m_1	0.54 kg
Concentrared mass at the end-effector	m_2	0.2 kg
Rayleigh damping constants	α	7×10^{-2} s^{-1}
	β	2.1×10^{-5} s

TABLE II
KINEMATIC AND DYNAMIC CHARACTERISTICS OF THE MANIPULATOR

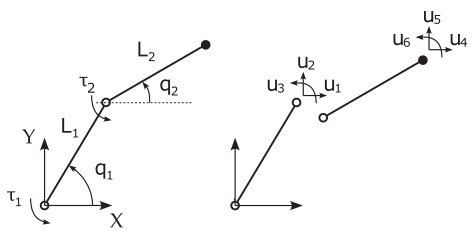

Fig. 1. Two link manipulator: rigid and elastic displacements

[32], which consists in iterating the numerical solution of the optimization problem by increasing at each iteration the value of σ, until a sufficient level of accuracy is obtained.

IV. NUMERICAL RESULTS

The mechanism taken as the testbench is a two-link planar RR manipulator with thin and long links. Actuation is provided by two electric motors. The kinematic and dynamic characteristics of the manipulator are reported in Table 2. Each link has a square section 1 cm wide, and the length of both links is 50 cm, and they are both made of aluminum. Therefore the mass of each link is around 130 g. The weight of actuators and of end-effector tool are included by using concentrated masses at the nodes of the mechanism.

Each flexible link has been represented by a single finite element, using 6 d.o.f. Euler-Bernoulli elements. The rigid displacements q_1 and q_2 and the six elastic displacements u_1, $u_2 \ldots u_6$ are shown in Figure 1.

The TPBVP problem stated in eq. (5) can be efficiently solved using the collocation method [32]. Matlab routine "bvp4c" has proved to be quite efficient for the task, therefore it has been used to obtain all the results presented in this work. The formulation of the problem requires some symbolical computations that will be briefly summarized here. First of all, an explicit form of the direct dynamics of the manipulator must be available as a vector with 18 rows:

$$\Omega(\mathbf{x}(t), \tau, t) \tag{9}$$

and the state vector:

$$\mathbf{x}(t) = [\dot{u}_1, \dot{u}_2, \ldots, \dot{u}_6, \dot{q}_1, \dot{q}_2, u_1, u_2, \ldots, u_6, q_1, q_2]^T \tag{10}$$

The control vector is $\tau = [\tau_1, \tau_2]^T$. The cost function of choice is:

$$f = \tau^T \mathbf{R} \tau + \mathbf{x}^T \mathbf{Q} \mathbf{x} + P(\mathbf{x}, \sigma) \tag{11}$$

$\mathbf{R}$ is a 2×2 matrix that weigh the value of the control input (i.e. the actuator torque), while $\mathbf{Q}$ is a 18×18 matrix that penalizes the various entries of the state vector $\mathbf{x}$. p is a cost function of the form (8). The Hamiltonian function (6) and the PMP necessary condition can be evaluated in their symbolic forms in a straightforward manner, in order to use them with the Matlab bvp4c routine.

A. Test I: constrained joint speed

The cost function used for this tests, which includes the penalty function on the angular speed of the second joint $\dot{q}_2$ is:

$$f = \tau^T \mathbf{R} \tau + \mathbf{x}^T \mathbf{Q} \mathbf{x} + W_p \left[p(\dot{q}_2 - \bar{\dot{q}}_2, \sigma) + p(-\dot{q}_2 - \bar{\dot{q}}_2, \sigma) \right] \tag{12}$$

W_p is a scalar weight than can be tuned together with σ in order to get a suitable approximation of the hard constraints on speed $\dot{q}_2$. The results shown in figures (2,3,4) have been evaluated using: $W_p = 100$, $\sigma = 8$ and $\bar{\dot{q}}_2 = 4\ rad/s$. The last one is the limit on the speed of the second joint of the robot. Matrices $\mathbf{R}$ and $\mathbf{Q}$ are used as weight on the control action and on the state $\mathbf{x}$ of the plant. The weights on the two torques are equal to 1, while weights on the speed of the joints are both equal to 0.5. Elastic displacements $\mathbf{u}$, their time derivative $\dot{\mathbf{u}}$ and the angular positions $\mathbf{q}$ are weighted with factors 5, 10 and 0.1 respectively. The use of a cost function with so many entries allows to tune in a very accurate manner the response of the system. Given the complexity of the problem, which is mostly due to the complexity of the dynamics involved, an effective trajectory planning procedure requires to the user to adopt a trial-and-error strategy for choosing the a suitable set of weighting factors. However the introduction of hard constraints should in some way simplify the procedure.

Figures 3-5 show the results of the trajectory planning algorithm for the case of a point-to-point trajectory with and without the use of constraints on the speed of the second joint of the robot. As it can be seen in Figures 2 and 3, the introduction of constraints on $\dot{q}_2$ has the effect of limiting below the chosen threshold the speed of the second joint, while the top speed of the first joint is increased by a 25 % factor. The resulting trade-off between the two peak joint speeds can represent an advantage in some circumstances. It is often found that it is convenient to use a lighter and less powerful motor at the second joint, and to chose an heavier and powerful one for the first joint, since it is connected to the chassis. In this case, it can be advantageous to reduce the actuation effort of the second motor with the technique introduced here. The trajectory of the end-effector of the robot (i.e. the tip of the second link of the robot) is quite different between the two cases of constrained and unconstrained solutions. As it can be seen in Figure 4, such difference is more evident during the

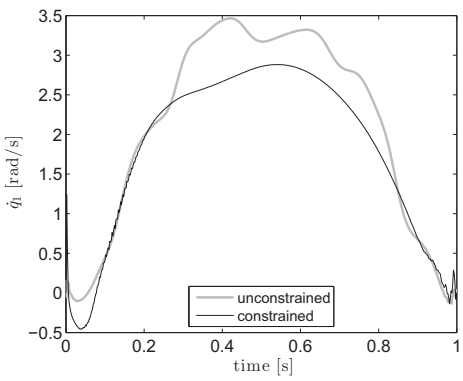

Fig. 2. Test I: time evolution of $\dot{q}_1$, constrained and unconstrained solution

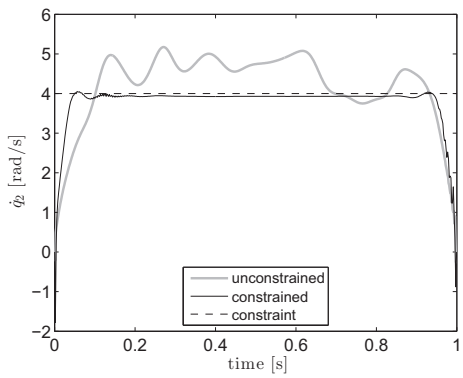

Fig. 3. Test I: time evolution of $\dot{q}_2$, constrained and unconstrained solution

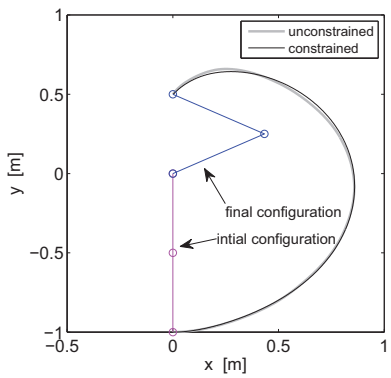

Fig. 4. Test I: trajecotry of the end effector in the operative space, constrained and unconstrained solutions

second half of the trajectory. Therefore we have shown that the use of constraints on joint speed does not worsen the accuracy of motion, or the smoothness of the resulting trajectory.

B. Test II: constrained elastic displacement

The cost function used to obtain constraints on elastic displacements is:

$$f = \tau^T \mathbf{R} \tau + \mathbf{x}^T \mathbf{Q} \mathbf{x} + W_p \left[p(u_6 - \overline{u}_6, \sigma) + p(-u_6 - \overline{u}_6, \sigma) \right] \tag{13}$$

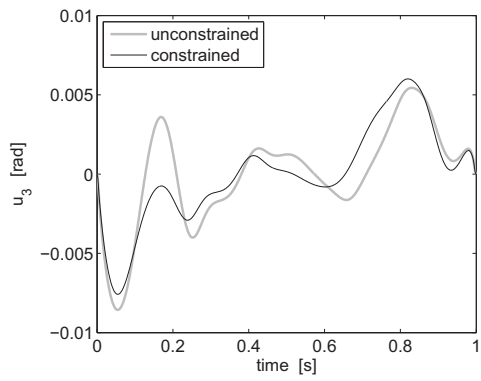

Fig. 5. Test II: time evolution of the elastic displacement u_3, constrained and unconstrained solution

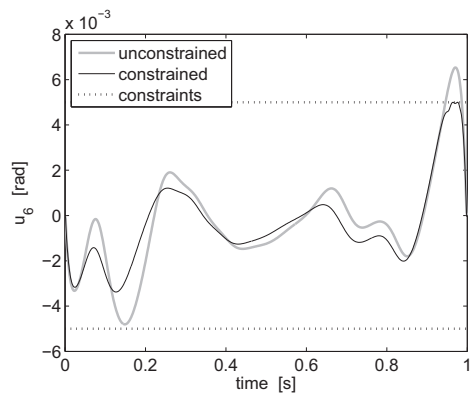

Fig. 6. Test II: time evolution of the elastic displacement u_6, constrained and unconstrained solution

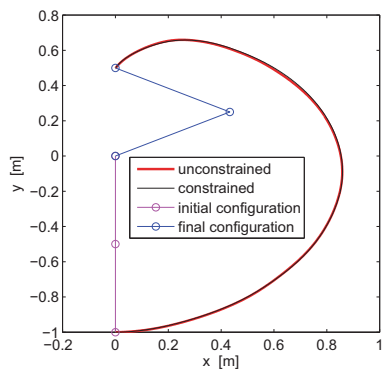

Fig. 7. Test II: trajectory of the end-effector, constrained and unconstrained solution

The results shown in figures (5,6,7) have been evaluated using: $W_p = \sigma = 10^4$ and $\overline{u}_6 = 5 \cdot 10^{-5}$. The latter is the constraint on the amplitude of the elastic displacement u_6. Matrices $\mathbf{R}$ and $\mathbf{Q}$ are tuned exactly as for Test I.

Figures 5, 6 shows the time evolution of the rotational elastic displacement at the end-point of the two links, i.e. u_3 and u_6, respectively. For the constrained solution, u_6 has been limited to the value of $5 \cdot 10^{-3}$ rad. As is can be seen

from in Figure 6, the introduction of the constraints allows to effectively include a bound on the elastic displacement at the end-effector. This happens without increasing the level of vibration acting on the first link of the robot, as visible in Figure 5. By looking at the time evolution of the elastic displacements of both links around time $t = 0.2$ s, we can also note that here we obtain also a notable reduction of elastic deformation. It should be also pointed out that the end-point of the trajectory is reached with null elastic displacement, null joint speed and and also the time derivative of the nodal elastic displacements, i.e. $\dot{\mathbf{u}}$ is equal to zero. This is an essential requirement for the accuracy of a rest-to-rest motion of a flexible mechanism. Figure 7 shows that the trajectory of the end-effector is basically not influenced by the introduction of constraints on elastic displacements.

V. Conclusion

In this paper the problem of planning of constrained trajectory for flexible-links mechanisms has been investigated. Unlike most of the available literature on the subject, dynamic modelling of FLMs is achieved trough the use of a nonlineare FEM-based approach, for maximum accuracy. The optimization problem is set as a nonlinear constrained optimization problem, which is translated into a two-point boundary value problem and solved numerically. Composite cost optimal with bounded joint speed and elastic deformation are achieved by introducing a novel penalty function. The capability of the proposed approach is tested here, by showing that the use of kinematic and dynamic constraints and a complex composite function can lead to the determination of trajectories with very limited vibrational phenomena. Numerical results are obtained for a planar two-link flexible mechanism, but the proposed approach can be easily adapted to other mechanisms as well

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