Robust model-based trajectory planning for nonlinear systems

P. Boscariol, A. Gasparetto

DIEGM - University of Udine, Via delle Scienze 206, 33100 Udine - Italy

Abstract

Model-based trajectory planning algorithms are capable of providing a high level of performance. However, they often lack in robustness, which severely limits their field of application. In this paper the method of parametric desensitization is applied to nonlinear models, providing a feasible solution to the problem of robust model-based trajectory planning for nonlinear plants with parametric uncertainties. By using an indirect variational solution method, the necessary optimality conditions deriving from the Pontryagin's minimum principle are imposed, and lead to a differential Two-Point Boundary Value Problem (TPBVP); numerical solution of the latter is accomplished by means of collocation techniques. The method is applied to two test-cases: a nonlinear spring-mass system and a flexible link manipulator with Coulombian friction. Results show that the technique developed in this paper can improve significantly the robustness of the resulting trajectory to parametric model mismatches in comparison with the conventional method.

Keywords: trajectory planning, robust, robotics

1. Introduction

The operation of high-speed manipulator requires the use of accurate and clever techniques both in the design of closed-loop controllers, both in the design of smooth trajectories. The first approach is aimed at ensuring that the robot follows a pre-defined trajectory, while the latter concerns the definition of the trajectory. The whole problem is even more evident and hard to solve when the robot presents some structural flexibility, since the presence of oscillations even after the completion of the task severely reduces the operativeness of the robot [17, 6, 21]. A large number of works have been developed to find solutions to the aforementioned problem by generating smooth trajectories, as reported in the review papers [3, 22]. A main distinction can be made among trajectory planning algorithms separating model-free and model-based approaches. The main advantage of the first approach is that the resulting trajectory can be adapted to several robots, and, moreover, the knowledge of the dynamics of the system is not needed. This is a very useful feature, since this kind of expertise is not

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often found in industry. On the other hand, model-based approach requires the knowledge of the dynamics of the robot for which the trajectory is planned, but generally they prove to be more accurate and usually lead to higher levels of performance. Most model-based approaches are based on optimal control theory, for this reason they provide a limited robustness to model-plant mismatches, as emphasized in [15]. This means that a trajectory that is optimal in the nominal case, is far from the optimal solution if applied to a perturbed plant.

Model-free approaches are often based on geometrical approaches [7, 18], since the trajectory is defined as a sequence of polynomial functions [10], splines [23] or NURBS [33]. Other possible ways to generate a trajectory is to use filters [8, 9].

Model-based approach have been studied in a large number of works, with applications to basically every kind of robot. The solution of the planning problem has been investigated for mobile robots in papers such as [5]. Flexible joint robot have been considered in [16, 32]. Also the design of trajectories for Flexible-Link Manipulators (FLM) have been studied quite extensively. Abe in [1] developed a method based on particle swarm optimization, while Kojima et. al [30] used a genetic algorithm to compute a trajectory for a two-link manipulator. A genetic algorithm is the choice for computing trajectories for flexible-link mechanisms in the work [4] by Ata et. al. Approaches based on the definition and the solution of Two-Point Boundary Value Problems (TPBVP) have been developed, among others, in [31, 11, 12] focusing on flexible link mechanisms.

In this work a solution to the problem of computing point-to-point trajectories is analyzed, with a particular focus on the robustness of the solution to parametric uncertainties. Therefore the method used here can be used when one or more parameters of the plant under consideration cannot be estimated with sufficient precision or when its change is due to an unmodeled dynamics. The topic of robustness have been extensively studied in the area of closed-loop control design [2, 28, 40], but to the best of authors' knowledge, there are few that specifically focus on robust trajectory planning algorithms. One example is [20], in which robustness is achieved by introducing in the fitness function a term of Gaussian cumulative noise.

The aim of this paper is to propose a method for planning a robust trajectory for mechatronic systems. Such method is based on the solution of a two-point boundary value problem and on the concept of parametric desensitization.

Moreover, the innovative method proposed here applies to nonlinear plants, therefore it greatly enhances the field of application of the method presented in [38, 29, 18], which is also based on the use of sensitivity functions. However, the mentioned method can be applied only to linear plants, and is used to design a closed-loop control system.

The capabilities of the approach analysed in this paper are shown by means of two benchmark problems, i.e. a single-mass undamped system with nonlinear elasticity, and a single-link manipulator with Coulombian friction.

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2. Formulation of trajectory planning algorithm as a two-point boundary value problem

The target here is to develop an optimal trajectory for a mechatronic system. Here point-to-point trajectory optimization problems are considered, in which only the initial and final end-effector positions are given, and the manipulator is free to move between them. The path is therefore subject to optimization, and it is selected with the aim of minimizing a cost functional. Such cost may depend on execution time, actuator effort, jerk (or torque rate), or a combination of these variables. First of all, let us define the optimization problem that represents the trajectory planning problem. Given a dynamic system, that might be linear or nonlinear, described by a differential equation in the form:

$$\dot{\mathbf{x}(t)} = \Pi(\mathbf{x}(t), t, \mathbf{u}) \tag{1}$$

in which $\mathbf{x}$ is the state variable of the system, and $\mathbf{u}$ is the control vector, and t is the time. If a cost function $f(\mathbf{x}, t, \mathbf{u})$ is chosen, the following optimization problem can be stated:

$$\begin{cases} \min J(\mathbf{x}(t), t, \mathbf{u}) = \min \int\limits_{t_0}^{t_f} f(\mathbf{x}, t, \mathbf{u}) dt \\ \textit{subject to}: \\ \mathbf{x}(t_0) = \alpha \\ \mathbf{x}(t_f) = \beta \\ \dot{\mathbf{x}}(t) = \Pi(\mathbf{x}(t), t, \mathbf{u}) \end{cases} \tag{2}$$

By solving this optimization problem a trajectory for the state vector $\mathbf{x}$ is found so that the cost function J is minimized. The trajectory is constrained respect to the dynamics of the system $\Pi(\mathbf{x}(t), t, \mathbf{u})$ and to the value of $\mathbf{x}$ at the initial ($t = t_0$) and final ($t = t_f$) time. A solution of the optimization problem in equation (2) can be found using the calculus of variations and Pontryagin's Minimum Principle (PMP) [36].

First of all, the Hamiltonian of the system must be defined as:

$$\mathcal{H} = f + \Lambda^T \Pi(\mathbf{x}(t), t, u) \tag{3}$$

in which $\Lambda = [\lambda_1, \ldots, \lambda_N]^T$ is the vector of Lagrangian multipliers, which has the same size of the state vector $\mathbf{x}$. The necessary conditions for finding a minimum of the problem in equation (2) are:

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = 0 \tag{4}$$

$$\dot{\mathbf{x}} = \frac{\partial \mathcal{H}}{\partial \Lambda} \tag{5}$$

$$\dot{\Lambda} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}} \tag{6}$$

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The above conditions can be put in a single system that makes the computation straightforward. By defining $\mathbf{u}^*$ as the solution of equation (4), $\mathcal{H}^*(\mathbf{x}, t)$ is the Hamiltionian in which $\mathbf{u}$ has been substituted with $\mathbf{u}^*$. A new system of ordinary differential equation can be defined as:

$$\dot{\mathbf{y}} = \begin{bmatrix} \dfrac{\partial \mathcal{H}^*}{\partial \Lambda} \\ \\ -\dfrac{\partial \mathcal{H}^*}{\partial \mathbf{x}} \end{bmatrix} \tag{7}$$

The new state vector $\mathbf{y}$ is obtained by augmenting the original state vector $\mathbf{x}$ with the vector of Lagrangian multipliers: $\mathbf{y} = [\mathbf{x}, \Lambda]^T$. Among the infinite possible trajectories of the dynamic system in equation (13), the aim is to find the one that obeys to the boundary conditions $\mathbf{y}(t_0) = \alpha$ and $\mathbf{y}(t_f) = \beta$.

A solution to this problem, that is basically a TPBVP (Two-Point Boundary Value Problem), could theoretically be found in closed form. In many cases, however, it is solved numerically, given the difficulty of finding an exact solution. Collocation method [37, 11] and shooting method [25] are often used for this task.

3. Formulation of the robust trajectory planning algorithm

The problem presented and solved in the previous section works very well when the dynamic model used for planning the trajectory can reproduce faithfully the actual dynamics of the real system. This does not happen in all situations, given the difficulty of describing a complex plant with a reasonably simple model. Moreover, sometimes it is not even possible to describe the dynamics of the plant with just a single model. A common situation is when a robot is driving a payload that changes, as in a pick & place operation. As the mass carried by the robotic manipulator changes, also its dynamic model is altered. Quite often also nonlinearities might be neglected during the modeling phase: in this case the trajectory planning algorithm and the control loop are required to compensate for the model-plant mismatches.

The solution to the problem of the robustness of trajectory planning algorithms proposed in this paper is based on the use of sensitivity functions, which are briefly introduced here. A function $\Omega(x, t, \eta)$ with continuous first partial derivatives with respect to x and η for all $(t, x, \eta) \in [t_0, t_1] \times R^n \times R^p$ is considered. It is also supposed that η_0 is the nominal value of the parameter η and that the differential equation:

$$\dot{x} = \Omega(t, x, \eta_0) \quad with\ x(t_0) = x_0$$

has unique solution $x(t, \eta)$ over $[t_0, t_1]$. The continuous differentiability of Ω with respect to x and η implies that the solution $x(t, \eta)$ is differentiable with respect to η near η_0. If one writes:

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$$x(t,\eta) = x_0 + \int_{t_0}^{t} \Omega(s, x(s,\eta), \eta)ds \tag{8}$$

then the partial derivatives of x with respect to η are:

$$x_\eta(t,\eta) = \int_{t_0}^{t} \frac{\partial \Omega}{\partial x}(s, x(s,\eta,\eta), x_\eta(s,\eta)) + \frac{\partial \Omega}{\partial \eta}(s, x(s,\eta), \eta)ds \tag{9}$$

where $x_\eta = [\partial x(t,\eta)/\partial \eta]$ and $[\partial x_0/\partial \eta] = 0$, since x_0 is independent of η.

By taking the derivative with respect to t, it can be seen that $x_\eta(t,\eta)$ satisfies the differential equation:

$$\frac{\partial x_\eta(t,\eta)}{\partial t} = A(t,\eta)x_\eta(t,\eta) + B(t,\eta) \tag{10}$$

where

$$A(t,\eta) = \left.\frac{\partial \Omega(x,t,\eta)}{\partial x}\right|_{x=x(t,\eta)}$$

and

$$B(t,\eta) = \left.\frac{\partial \Omega(x,t,\eta)}{\partial \eta}\right|_{x=x(t,\eta)}$$

Let $S(t) = x_\eta(t,\eta)$, then $S(t)$ is the unique solution of the equation:

$$\dot{S}(t) = A(t,\eta)S(t) + B(t,\eta) \tag{11}$$

$S(t)$ is the sensitivity function and eq. (11) is the sensitivity equation. The sensitivity function allows to estimate the effect of parameter variations on the solution of eq. (8).

The main idea behind the technique used in this paper is to augment the plant dynamic model with the partial derivatives of the ODE system with respect to a parameter η of choice. These partial derivatives are called sensitivity functions. By imposing that their values must be zero at a given point of a trajectory, the robustness with respect to the parameter η is increased. The effectiveness of this approach has been shown both numerically [24, 38], and experimentally [29] but only for the design of closed-loop control systems for linear systems. For this reason the procedure followed in the aforementioned papers cannot be applied to the test cases used here, which involve nonlinear systems.

First of all, let us take into consideration the system of ordinary differential equations $\Pi(\mathbf{x}, t, \mathbf{u}, \eta)$ that describe the dynamics of the plant under consideration. For each equation of the system, a sensitivity function can be written using the notation of eq. (11). By taking the partial derivative of each equation that belongs to $\Pi(\mathbf{x}, t, \mathbf{u}, \eta)$ with respect to the uncertain parameter η, a new set of differential equations are found in the form:

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$$\dot{\mathbf{S}}(t) = \mathbf{A}(t,\eta)\mathbf{S}(t) + \mathbf{B}(t,\eta) \tag{12}$$

Now an augmented system of differential equations can be composed by joining equation (1) with the system in equation (12):

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{S}}(t) \end{bmatrix} = \begin{bmatrix} \Pi(\mathbf{x}(t), t, \mathbf{u}, \eta) \\ \mathbf{A}(t,\eta)\mathbf{S}(t,\eta) + \mathbf{B}(t,\eta) \end{bmatrix} \tag{13}$$

It should be highlighted that the definition of the sensitivity equations allows to calculate in a straightforward manner eq. (13), since:

$$\mathbf{S}(t) := \frac{\partial \mathbf{x}(t)}{\partial \eta}; \quad \dot{\mathbf{S}}(t) = \frac{d}{dt}\frac{\partial \mathbf{x}(t)}{\partial \eta}; \quad \mathbf{A}(t,\eta)\mathbf{S}(t,\eta) + \mathbf{B}(t,\eta) = \frac{\partial \Pi(\mathbf{x}(t), t, \mathbf{u})}{\partial \eta} \tag{14}$$

In the cases under consideration here the uncertain parameter is just one, η, but the method shown here allows to take into consideration an arbitrary number of uncertain parameters. If $\mathbf{x}(t) \in \Re^n$, and there are m uncertain parameters, than simply $\mathbf{S}(t) \in \Re^{nm}$.

Now the optimization problem in equation (2) can be reformulated by including the sensitivity conditions as well:

$$\begin{cases} \min J(\mathbf{x}(t), \mathbf{S}(t), t, \mathbf{u}) = \min \int\limits_{t_0}^{t_f} f(\mathbf{x}, \mathbf{S}, t, \mathbf{u}) dt \\ s.to. \\ \mathbf{x}(t_0) = \alpha \\ \mathbf{x}(t_f) = \beta \\ \mathbf{S}(t_0) = 0 \\ \mathbf{S}(t_f) = 0 \\ \dot{\mathbf{x}}(t) = \Pi(\mathbf{x}(t), t, \mathbf{u}) \\ \dot{\mathbf{S}}(t) = \mathbf{A}(t,\eta)\mathbf{S}(t,\eta) + \mathbf{B}(t,\eta) \end{cases} \tag{15}$$

The difference between equation (2) and (15) is that the latter problems include a larger number of constraints. As it will be shown in the following, by imposing that the sensitivity function are equal to zero at the beginning and at the end of the trajectory, the parametric robustness of the planned trajectory can be improved.

It is worthwhile to point out that optimization problem (15) can be also computed automatically using a Computer Algebra System, such as Sage [39] or Axiom [27], thus making the formulation of the TPBVP completely automatic once Π, α, β, t_f and η are chosen. The same numerical method used for solving the problem in eq. (2) can be used here.

4. Test-case I: nonlinear mass-spring system

The test case under consideration here is a single mass system with a nonlinear spring, as in Figure 1. No damping elements are introduced. This bench-

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mark problem can represents a wide range of physical systems which are characterized by the low-frequency a low-damping nonlinear dynamics such as gantry cranes [13, 14], tanks with slosh [26] or tape drives [35], just to cite a few examples. The nonlinear characteristic of the spring is described by the elastic force F:

$$F = kq + kq^3 \tag{16}$$

being q the displacement of the mass m from the rest position. Therefore if u is the external force applied to the mass, the dynamics of the system is described by the second-order differential equation:

$$m\ddot{q} = -kq - kq^3 + u \tag{17}$$

The second-order ODE in equation (17) can be written in its first-order version by choosing the state vector $\mathbf{x}$ as $\mathbf{x} = [\dot{q}, q]^T$. With this choice the ODE system that will be used to compute the optimal trajectory is:

$$\dot{\mathbf{x}} = \begin{bmatrix} -\dfrac{k}{m}(q + q^3) + \dfrac{u}{m} \\ \dot{q} \end{bmatrix} \tag{18}$$

The system in eq. (18) can be augmented by including the two sensitivity function of the state vector $\mathbf{x}$ with respect to the elastic constant k, according to the notation of equation (11):

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \dot{q}}{\partial k}\right) &= -\frac{1}{m}(q + q^3) - \frac{k}{m}\frac{\partial q}{\partial k}\left(1 + 3q^2\right) \\ \frac{d}{dt}\left(\frac{\partial q}{\partial k}\right) &= \frac{\partial \dot{q}}{\partial k} \end{aligned} \tag{19}$$

In this formulation the vector of sensitivity functions is $\mathbf{S}(t) = \left[\dfrac{\partial \dot{q}}{\partial k}, \dfrac{\partial q}{\partial k}\right]^T$. Therefore the TPBVP must be formulated considering as the plant dynamics the augmented ODE:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{S}}(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k}{m}(q + q^3) + \dfrac{u}{m} \\ \dot{q} \\ -\dfrac{1}{m}(q + q^3) - \dfrac{k}{m}\dfrac{\partial q}{\partial k}\left(1 + 3q^2\right) \\ \dfrac{\partial \dot{q}}{\partial k} \end{bmatrix} \tag{20}$$

Now the number of ODE is four, therefore the application of the PMP requires to use four Lagrangian multipliers.

Figure 2 shows the planned trajectory for the nominal and the robust case, in terms of mass position q. The boundary conditions for the two TPBVP solved are: $q(t = 0) = 1$, $\dot{q}(t = 0) = 0$, $q(t = t_f) = 0$, $\dot{q}(t = t_f) = 0$, and the final time

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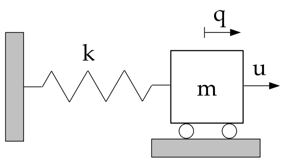

Figure 1: Mass-spring system

is $t_f = 2$ s. Therefore the task represents a rest-to-rest maneuver with fixed final time.

In order to evaluate the actual robustness of the two trajectories, Figure 3 and 4 shows the evolution of the mass position q when a perturbed plant is fed with a feedforward force signal computed using the two strategies explained above. It should be made clear that the application of a feedforward force profile is used just to provide an evaluation of the residual vibration after the motion completion, since our aim is to plan a trajectory for the mass position in the interval $t \in [0, t_f]$.

It can be clearly seen that for a value of the elastic constant k equal to 1.4 N/m (Figure 3) and $k = 0.6$ N/m (Figure 4) the residual vibration are much lower for the robust trajectory. In the second case, the amplitude of the peak residual vibration is close to zero. Both trajectories have been computed using the nominal value $k = 1$ N/m.

The control profile u^* is the solution of the first necessary condition imposed by Pontryagin principle, i.e. eq. (4). For both the nominal and the robust solution the cost function of choice is $J = u^2/2$. The application of the definition (3) to the ODE system of eq. (20) leads to the Hamiltonian:

$$\mathcal{H} = \frac{u^2}{2} - \lambda_1 \left(\frac{k}{m}(q + q^3) - \frac{u}{m} \right) + \lambda_2 \dot{q} - \frac{\lambda_3}{m} \left(q + q^3 - k\frac{\partial q}{\partial k}(1 + 3q^2) \right) + \lambda_4 \frac{\partial \dot{q}}{\partial k} \tag{21}$$

and therefore, according to (4):

$$u^* = -\frac{\lambda_1}{m} \tag{22}$$

The control profile u^* is in general a function of $\mathbf{x}(t)$, $\mathbf{S}(t)$ and $\Lambda(t)$ and therefore it can be evaluated once the numerical solution to the problem in eq. (15) is solved ([37]). The same procedure also applies to the solution of the nominal problem of eq. (2). The value of u^* for the nominal and the robust case are shown in figure 5. It can be seen that the introduction of the robustness constraints does lead, in this case, to an higher level of actuator effort.

A more detailed evaluation of the robustness is provided in Figure 6: here the amplitude of the peak residual vibration is plotted versus the elastic constant k, which varies in the range $[0.3, 1.7]$ N/m, thus embracing a $\pm 70\%$ variation. The result is that the robust trajectory performs better than the nominal one for all the values of k, retaining the performance of the nominal trajectory for

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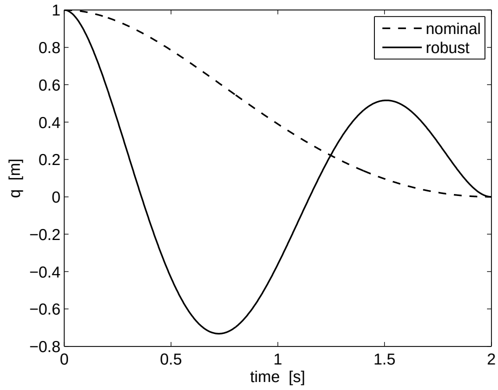

Figure 2: Nominal and robust trajectory for nonlinear spring-mass system: mass position q

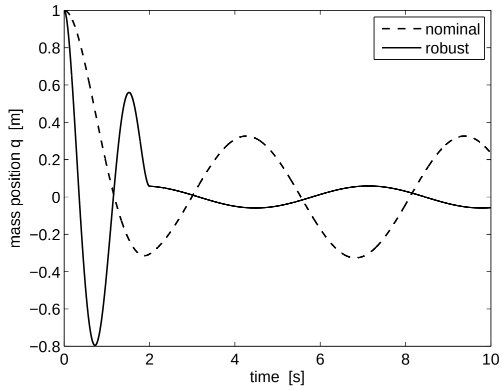

Figure 3: Response of the system with $k = 1.4$ N/m: nominal and robust trajectories

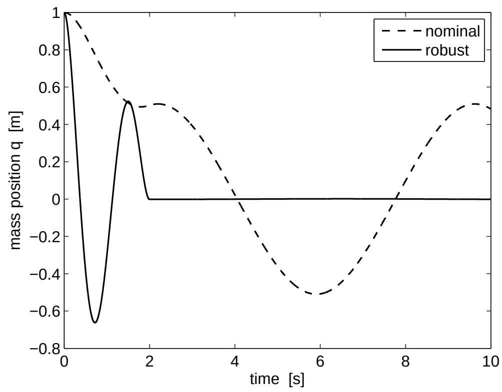

Figure 4: Response of the system with $k = 0.6$ N/m: nominal and robust trajectories

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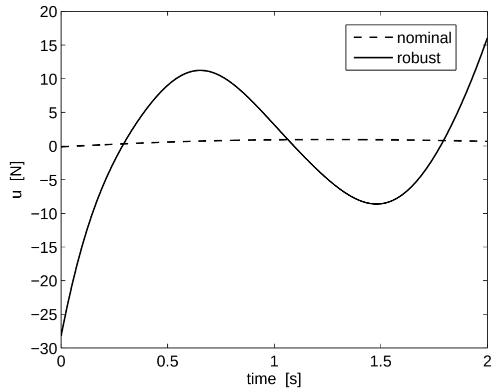

Figure 5: Mass-spring system: nominal and robust control action

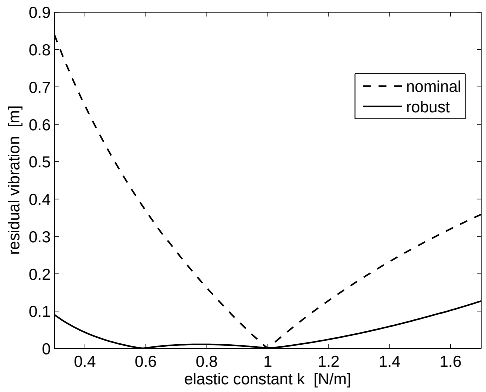

Figure 6: Nonlinear spring-mass system: peak residual vibration vs. elastic constant k

$k = 1$ N/m. Null residual vibration is also obtained for $k \approx 0.6$ N/m for the robust trajectory.

The technique used in this work is based on the use of constraints on the value of the sensitivity function at the boundaries of the trajectory. In particular, by imposing that the sensitivity functions of eq. (19) must be zero at final time, a minimization of the sensitivity of the trajectories to the elastic constant k is obtained. In other words, a maximization of the robustness of the trajectory is obtained. Since the additional constraint is posed at final time t_f, what is obtained is not the robustness of the whole trajectory to parametric uncertainties, but the robustness at final time. The first result could be obtained by modifying the cost function J of eq. (2) or by adding a constraint to it. This is not the case taken into consideration in this work, since our purpose is to achieve minimal residual vibration at the end of the trajectory. Since robustness is inversely

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proportional to sensitivity, the robustness properties of the planned trajectories can be evaluated and shown with greater detail by a sensitivity analysis. Again, this sensitivity analysis is performed for the final point of the trajectory.
The total energy of the mass-spring system at time t is:

$$E(t) = T(t) + U(t) \tag{23}$$

being T the kinetic energy and U the elastic energy. The first one can be evaluated as:

$$T(t) = \frac{1}{2}\dot{q}(t)^2 \tag{24}$$

while the elastic energy is:

$$U(t) = \int_0^{q^*} F dq = \int_0^{q^*} \left(kq + kq^3\right) dq = k\left(\frac{1}{2}{q^*}^2 + \frac{1}{4}{q^*}^4\right) \tag{25}$$

The total energy of the system at final time can be evaluated numerically for both the nominal and the robust trajectory reported above. The ratio $\frac{\partial E}{\partial k}\big|_{t=t_f}$ is the sensitivity of the total energy of the system at final time. The value of the energy of the system is used to estimate the error introduced by the parametric uncertainty on the stiffness k, since the final energy of the system should be zero in the rest-to-rest motion under consideration here. The value of this sensitivity is shown in Figure 7 as a function of the elastic constant k: it can be clearly seen that within the range of k under consideration, the absolute value of the sensitivity for the robust trajectories is always smaller than for the nominal trajectory. This clearly indicates that the robustness, intended in the sense of achieving lower residual vibration, is increased by the proposed method in comparison with the conventional optimization method of eq. (2).
It can be demonstrated that the minimization of the sensitivity of residual energy is a direct consequence of the choice of the additional constraints of the optimization problem of eq. (15). The residual energy of the system, i.e. the total energy of the system at the final time of the trajectory is, according to eq. (19–20) directly proportional to the mass displacement $q(t_f)$ and the mass speed $\dot{q}(t_f)$. Equivalently, the residual energy of the system is directly proportional to the distance between the actual final state vector $\mathbf{x}(t_f)$.
The method introduced here includes a constraint on the sensitivity of the state vector at the final time of the planned trajectory, with the direct effect of minimizing the sensitivity of the state vector $\mathbf{x}(t_f)$. Since the minimization of the sensitivity to the uncertain parameter k implies maximum robustness to the variation of the same parameter, and given the direct dependence of the total final energy to the state vector at final time, it can be inferred that the minimization of the final state vector implies the maximization of the robustness in terms of residual energy of the system.
This statement holds true in the neighborhood of the nominal value of the uncertain parameter, since the definition of the sensitivity function is, as explained in section 3, limited to this range.

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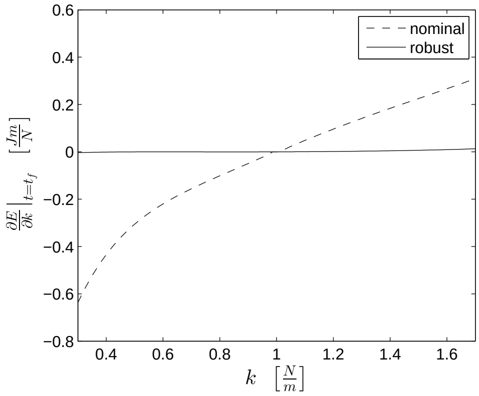

Figure 7: Mass-spring mechanism: sensitivity of residual energy to elastic constant k, robust and nominal trajectory

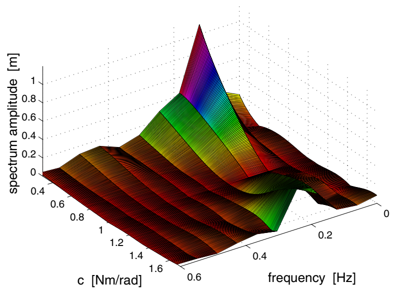

Figure 8: Mass-spring mechanism: frequency spectrum of residual vibration vs. elastic constant k, nominal trajectory

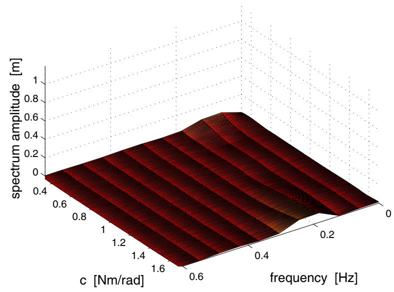

Figure 9: Mass-spring mechanism: frequency spectrum of residual vibration vs. elastic constant k, robust trajectory

In order to further quantify the effect of the application of the proposed method in terms of residual vibration reduction, the amplitude spectrum of residual vibration is shown for values of k in the range $[0.3, 1.7]$ N/m for the nominal and robust trajectory in Figure 8 and Figure 9, respectively. It can be seen that the use of a robust trajectory leads to a lower harmonic content for every frequency and value of spring stiffness taken into consideration.

5. Test-case II: flexible-link mechanism

The test case considered in this section is a single-link very flexible mechanism. The model proposed, which has been experimentally validated in [19] and used, among others, in [34] will be briefly recalled here. The model is valid under the assumption that link mass is concentrated at the tip, and that the mechanism rotate on a horizontal plane, so that gravity effects can be neglected.

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The tip mass is m, while the length of the link is L, and the elastic constant of the link is c. The mechanism is actuated by a DC motor with a reduction gear whose reduction ratio is n, and the inertia of the motor shaft is J_m. The motor dynamics can be described by the equation:

$$ku = J_m \ddot{\hat{\theta}}_m + v\dot{\hat{\theta}}_m + \hat{\Gamma}_c + \hat{\Gamma}_{coup} \tag{26}$$

in which k is the electromechanical constant of the motor, u is the DC voltage applied to the motor, v is the viscous friction coefficient and $\hat{\Gamma}_c = \mu_r sign(\dot{\hat{\theta}}_m)$ is the Coulomb friction acting on the motor. $\hat{\Gamma}_{coup}$ is the coupling torque between the motor and the link. Being θ_m the angular position of the motor, and θ_t the angular position of the tip mass, the coupling torque can be measured as:

$$\Gamma_{coup} = mL^2\ddot{\theta}_m = c(\theta_m - \theta_t) \tag{27}$$

The quantities indicated with a 'hat' mark are meant as measured on the motor shaft, while the ones without are measured on the global reference frame {X,Y} as in Figure 10. Therefore the following can be used: $\dot{\theta} = \dot{\hat{\theta}}/n$ and $\Gamma = n\hat{\Gamma}$, leading to:

$$\hat{\Gamma}_{coup} = \frac{c}{n}(\theta_m - \theta_t) \tag{28}$$

Equations (26) and (28) can be used together to define the dynamics of the whole systems as:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\theta}_m \\ -\frac{v}{J_m}\dot{\theta}_m - \frac{1}{J_m n^2}\Gamma_{coup} + \mu_r sign(\dot{\theta}_m) + \frac{k}{J_m n}u \\ \dot{\Gamma}_{coup} \\ -\frac{cv}{J_m}\dot{\theta}_m - \frac{c}{J_m n^2 + mL^2}\Gamma_{coup} + \mu_r sign(\dot{\theta}_m) + \frac{ck}{J_m n}u \end{bmatrix} \tag{29}$$

in which $\dot{\mathbf{x}} = [\theta_m, \dot{\theta}_m, \Gamma_{coup}, \dot{\Gamma}_{coup}]^T$. The ODE system described by equation (29) is nonlinear, due to the presence of the Coulomb friction term $\mu_r sign(\dot{\theta}_m)$. The value of the parameters that appear in eq. (29) are shown in Table 1.

The ODE system in equation (29) can be used to plan both a nominal and a robust trajectory. In the latter case, the ODE system is augmented with the partial derivatives evaluated with respect to the elastic constant c, therefore increasing the robustness to this parameter.

The four sensitivity functions of the ODE system in (29) are evaluated by taking its partial derivatives with respect to c:

$$S_1(\mathbf{x}, t) = \frac{\partial \dot{\theta}_m}{\partial c} \tag{30}$$

$$S_2(\mathbf{x}, t) = \mu_r sign\left(\frac{\partial \dot{\theta}_m}{\partial c}\right) - \frac{\partial \Gamma_{coup}}{\partial c}\frac{1}{J_m n^2} + \frac{v}{J_m}\frac{\partial \dot{\theta}_m}{\partial c} \tag{31}$$

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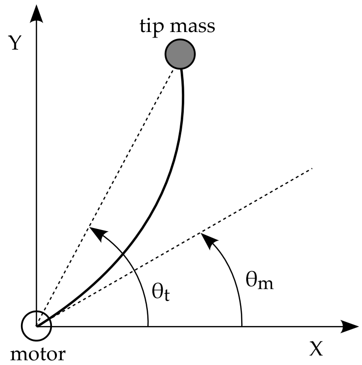

Figure 10: A flexible single-link mechanism

parameter	*symbol*	*value*
motor inertia	J_m	$2 \cdot 10^{-4}\ kgm^2$
electric motor constant	k	$0.3\ Nm/V$
reduction ratio	n	50
viscous friction constant	v	$1 \cdot 10^{-3}\ Nm/s$
Coulomb friction constant	μ_r	$0.5\ V$
tip mass	m	$0.3\ kg$
link length	L	$0.7\ m$
link elastic constant	c	$5\ Nm/rad$

Table 1: Flexible-link mechanisms: model parameters

$$S_3(\mathbf{x}, t) = \frac{\partial \dot{\Gamma}_{coup}}{\partial c} \tag{32}$$

$$\begin{aligned} S_4(\mathbf{x}, t) = \mu_r sign\left(\frac{\partial \dot{\theta}_m}{\partial c}\right) + \Gamma_{coup}\left(\frac{1}{Jn^2} + \frac{1}{L^2 m}\right) + \\ + \frac{\partial \Gamma_{coup}}{\partial c}\left(\frac{c}{J_m n^2} + \frac{c}{L^2 m}\right) - \frac{v}{J_m}\dot{\theta}_m - \frac{cv}{J_m}\frac{\partial \dot{\theta}_m}{\partial c} \end{aligned} \tag{33}$$

The two trajectories are shown in Figure 11. Both are evaluated using the cost function $J = \frac{1}{2}u^2 + \frac{1}{2}(\theta_m - \theta_t)^2$, therefore the resulting trajectory is minimum torque-minimum vibration.

The response of the system for $c = 4$, $c = 5$ and $c = 6$ is shown in Figure 13 for the nominal trajectory. It can be seen that zero residual vibration is obtained only for the nominal value ($c = 5$) of the elastic constant, and that large residual vibrations occur for both $c = 4$ and $c = 6$. The same applies

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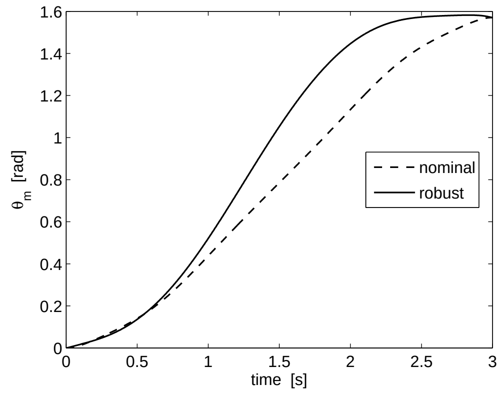

Figure 11: Single-link mechanism: nominal and robust trajectory

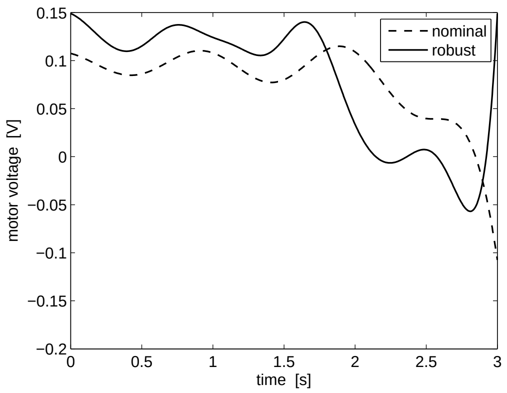

Figure 12: Single-link mechanism: nominal and robust control action

to the results shown in Figure 14: again zero residual vibration is obtained for $c = 5$, but for $c = 6$ and $c = 4$ the amplitude of the residual vibration is reduced in comparison to the use of the nominal trajectory. The peak amplitude of residual vibration is shown in Figure 15 as a function of the elastic constant c in the range $c \in [1, 9]$. It can be seen that the robust trajectory performs better, in terms of residual vibration for all the values of c taken into consideration. All the mentioned plots have been obtained by using the optimal control profiles shown in figure 12.

Again, the improvement of the robustness properties can be evaluated trough the use of the sensitivity of residual energy to the value of the elastic constant c. The total energy of the system is:

$$E(t) = T(t) + U(t) \tag{34}$$

in which:

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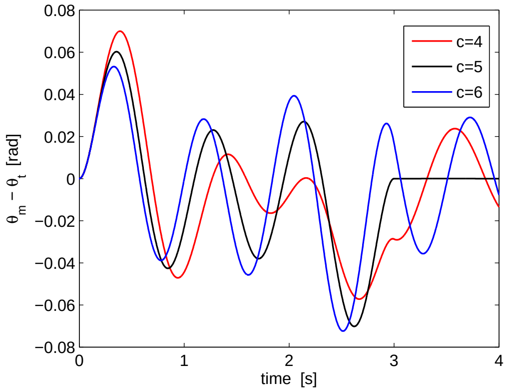

Figure 13: Single-link mechanism: elastic displacement $\theta_m - \theta_t$, nominal trajectory with $c = 4$ Nm/rad,$c = 5$ Nm/rad and $c = 6$ Nm/rad

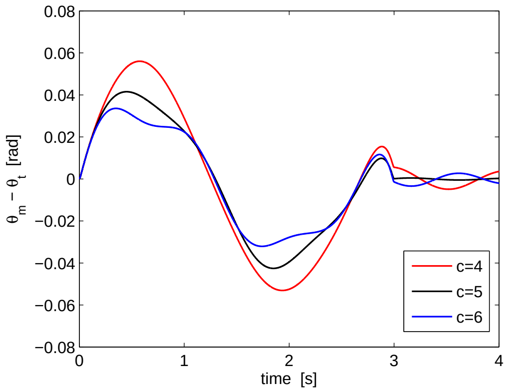

Figure 14: Single-link mechanism: elastic displacement $\theta_m - \theta_t$, robust trajectory with $c = 4$ Nm/rad,$c = 5$ Nm/rad and $c = 6$ Nm/rad

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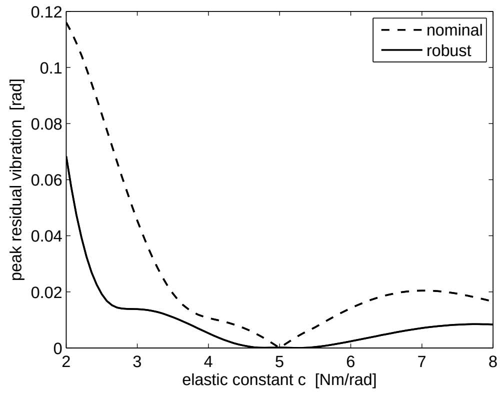

Figure 15: Single-link mechanism: peak residual vibration vs. elastic constant c

$$T(t) = \frac{1}{2} J_m \dot{\theta}_m(t)^2 + \frac{1}{2} m L^2 \dot{\theta}_t(t)^2 \tag{35}$$

and

$$U(t) = \frac{1}{2} c \left(\theta_m(t) - \theta_t(t)\right)^2 \tag{36}$$

The sensitivity function under investigation here is the partial derivative of the total energy of the system with respect to the link stiffness c, i.e. $\frac{\partial E}{\partial c}\Big|_{t=t_f}$. The values of the sensitivity function is shown in Figure 16 for the range of c between 2 and 8 Nm/rad. It can be seen that the absolute value of the sensitivity coefficient evaluated for the robust trajectory is always smaller than the same quantity evaluated for the nominal trajectory. This result is the same found for the case of the spring-mass system considered in the previous section. Therefore also here it can be inferred that the procedure used in this work can enhance parametric robustness in the sense of residual energy in a rest-to-rest motion. In this case, similarly to the previous test-case, the minimization of the sensitivity of the state vector $\mathbf{x}(t_f)$ also implies the maximum robustness in the sense of residual energy. According to eq. (30-32) the minimum residual energy is obtained for null values of $\theta_m - \theta_t$, $\dot{\theta}_m$ and $\dot{\theta}_f$. This values correspond to the rest conditions to be achieved at the end of the trajectory: this implies that minimal sensitivity at final time implies minimal sensitivity, thus maximum robustness, of the residual total energy. This evidence is also shown numerically by the analysis of the function $\frac{\partial E}{\partial c}\Big|_{t=t_f}$ whose numerical value is shown in Figure 16.

The results of a spectral analysis on the residual vibrations for several values of c in the range $[2, 8]$ Nm/rad are shown in Figure 17 and 18, respectively. The graph in Figure 18 has a less pronounced peak than the graph in Figure 17, meaning that the worst-case behavior of the robust trajectory is improved by

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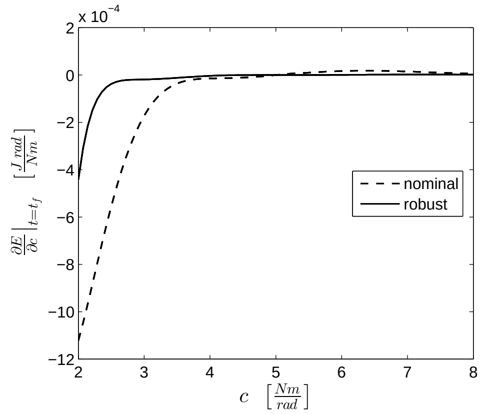

Figure 16: Single link mechanism: sensitivity of residual energy to elastic constant c, robust and nominal trajectory

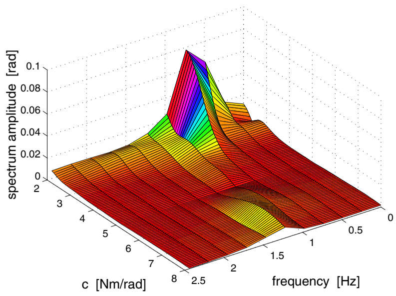

Figure 17: Single-link mechanism: frequency spectrum of residual vibration vs. elastic constant c, nominal trajectory

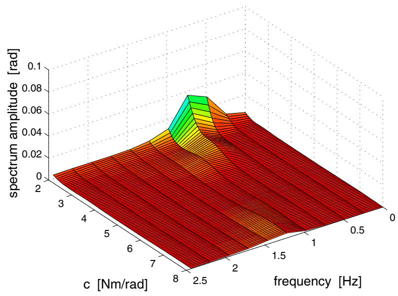

Figure 18: Single-link mechanism: frequency spectrum of residual vibration vs. elastic constant c, robust trajectory

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the robust technique introduced in this paper. Also the "flatter" behavior of the graph in Figure 18 around the nominal value of c clearly indicates the reduced sensitivity of the residual vibration spectral content produced by the robust trajectory. This features can be effectively used to improve the performance of the system in the cases that the elastic constant c cannot be estimated with precision, or in the case when it is perturbed by an unmodeled behavior.

6. Conclusions

In this paper the problem of model-based trajectory planning of mechanisms is dealt with. The paper introduces a method to improve the robustness of the resulting trajectory to parametric uncertainties. The method is based on the use of sensitivity functions. Unlike previous works, the trajectory planning algorithm presented here applies also to nonlinear plants. The improved robustness is shown in terms of residual vibration in a rest-to-rest motion for an undamped system, represented by a nonlinear mass-spring model, and for a lightly damped nonlinear model, such as a very-flexible single link mechanism with Coulombian friction. In both cases improved robustness is achieved by the method presented here. In particular, a minor level of residual vibration is achieved for all values of elastic constant for the mass-spring system and the very-flexible single-link mechanism.

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