Evolution of a Dynamic Model for Flexible Multibody Systems

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Abstract In this paper the evolution of a dynamic model for flexible multibody systems is presented. This model is based on an equivalent rigid-link system (ERLS) and, in the first formulation, has been exploited together with a FEM approach for the modeling of planar flexible-link mechanisms. Subsequently, the model has been linearized in order to be applied for control purposes and then it has been extended to the three-dimensional case. In the last years, a modal approach has been developed and the ERLS concept has been applied in order to formulate the dynamics of spatial flexible mechanisms with a component mode synthesis (CMS) technique.

Keywords Dynamic model · Flexible multibody system · Equivalent rigid-link system · Linearization · Component mode synthesis

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1 Introduction

In the last 20 years, the demand for high speed operations of mechatronic systems has pushed the study of dynamic models and controllers for flexible multibody systems. An approach to model multibody dynamics is based on a rigid-body model of the mechanism, to which elastic deformations are added to take link flexibility into account: this yields a coupled set of non-linear partial differential equations. In order to obtain a finite-dimensional problem formulated by a set of ordinary differential equations from these partial differential equations, two approaches have been proposed in the literature, namely the "nodal" and the "modal" approach [7, 8, 14, 18–22, 24, 31].

Although very popular, the latter approach has the drawback to yield a system of coupled differential equations with no separation between the rigid-body motion and the elastic deformation of the flexible body. The authors of this paper carried out, throughout a period of almost 30 years, extensive research that led to the formulation and experimental validation of a dynamic model based on the nodal approach.

The model is based on the concept of Equivalent Rigid-Link System (ERLS), first introduced in [6, 26, 27]. The first studies and the original formulation of the model (2D case) were done by Giovagnoni and Rossi in the 1980s [15, 16]. Giovagnoni validated the model for a 4-link flexible mechanism in 1994 [17]. Gasparetto validated the model for a 5-link flexible mechanism [11], linearized the original model [10] and used the model to test some controllers [1–5, 9, 12, 23, 25, 32]. Vidoni et al. [28, 29] extended the model to the 3D case and developed an efficient simulator of flexible multibody systems based on the extended model [13]. Lately, the ERLS principle was used to develop a modal approach to the dynamic modelling of flexible multibody systems [30].

2 The Original Dynamic Model

The dynamic model of flexible link multibody systems was originally developed for planar mechanisms [15, 16]. Every link is divided into finite elements, and the elastic displacements are defined with respect to an Equivalent Rigid Link Mechanism (ERLS), as shown in Fig. 1.

In Fig. 1, $\vec{u}$ is the nodal displacement vector and the vector $\vec{r}$ contains the positions of the nodes belonging to the ERLS. The vector $\vec{p}$ of the position of the generic point of the finite element is given by adding the vector $\vec{w}$ of the position of the corresponding point in the ERLS to the elastic displacement $\vec{v}$:

$$\vec{p} = \vec{w} + \vec{v} \tag{1}$$

Similarly, the displacements and the rotations at the nodes are given by the sum of the ERLS position and the elastic displacements:

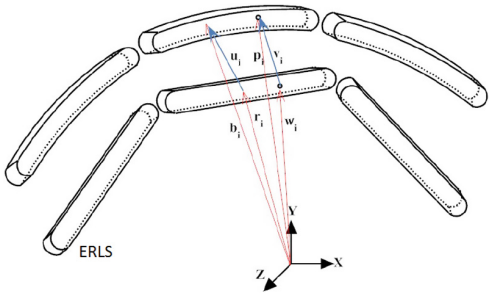

Fig. 1 Model of the dynamic system

$$\vec{b} = \vec{u} + \vec{r} \tag{2}$$

Similar relations hold for the infinitesimal displacements $d\vec{p}$ and $d\vec{b}$.

The position, velocity and acceleration of the ERLS are functions of the vector $\vec{q}$ of the free coordinates:

$$d\vec{r} = \vec{S}(\vec{q})d\vec{q} \tag{3}$$

$$\dot{\vec{r}} = \vec{S}(\vec{q})\dot{\vec{q}} \tag{4}$$

$$\ddot{\vec{r}} = \vec{S}(\vec{q})\ddot{\vec{q}} + \dot{\vec{S}}(\vec{q}, \dot{\vec{q}})\dot{\vec{q}} = \vec{S}(\vec{q})\ddot{\vec{q}} + \left(\sum_k \dot{q}_k \frac{\partial \vec{S}}{\partial q_k}\right) \tag{5}$$

where $\vec{S}(\vec{q})$ is the matrix of the sensitivity coefficients for all the nodes. Once the kinematics has been defined, the dynamic equations of motion for the flexible mechanism can be obtained by applying the principle of virtual work:

$$dW^{inertia} + dW^{elastic} + dW^{external} = 0 \tag{6}$$

From Eq. (6), according to [5], two dynamic equations of motion can be obtained:

$$d\vec{u}^T \vec{M}(\ddot{\vec{r}} + \ddot{\vec{u}}) + 2d\vec{u}^T \vec{M}_G \dot{\vec{u}} + d\vec{u}^T \vec{K}\vec{u} = d\vec{u}^T(\vec{f}_g + \vec{f}) \tag{7}$$

$$d\vec{r}^T \vec{M}(\ddot{\vec{r}} + \ddot{\vec{u}}) + 2d\vec{r}^T \vec{M}_G \dot{\vec{u}} + d\vec{u}^T \vec{K}\vec{u} = d\vec{r}^T(\vec{f}_g + \vec{f}) \tag{8}$$

where $\vec{M}$ is the mass matrix, $\vec{M}_G$ the Coriolis matrix, $\vec{K}$ the stiffness matrix of the mechanism; $\vec{f}_g$ is the gravity vector and $\vec{f}$ the vector of the external loads applied to the mechanism. Equation (7) formulates the nodal equilibrium, namely equivalent loads applied to every node must be in equilibrium. Equation (8) formulates the overall equilibrium, namely for any virtual displacement of the ERLS all the equivalent nodal loads produce no work. By expressing the infinitesimal displacements of the ERLS in terms of the sensitivity coefficient matrix, as in Eq. (3), the $d\vec{u}$'s and the $d\vec{r}$'s can be cancelled from Eqs. (7) and (8), thus obtaining:

$$\vec{M}(\ddot{\vec{r}} + \ddot{\vec{u}}) + 2\vec{M}_G \dot{\vec{u}} + \vec{K}\vec{u} = (\vec{f}_g + \vec{f}) \tag{9}$$

$$d\vec{S}^T \vec{M}(\ddot{\vec{r}} + \ddot{\vec{u}}) + 2\vec{S}^T \vec{M}_G \dot{\vec{u}} = \vec{S}^T(\vec{f}_g + \vec{f}) \tag{10}$$

which in matrix form can be written as:

$$\begin{bmatrix} \vec{M} & \vec{M}\vec{S} \\ \vec{S}^T\vec{M} & \vec{S}^T\vec{M}\vec{S} \end{bmatrix} \begin{bmatrix} \ddot{\vec{u}} \\ \ddot{\vec{q}} \end{bmatrix} = \begin{bmatrix} \vec{t}(\vec{u},\dot{\vec{u}},\vec{q},\dot{\vec{q}}) \\ \vec{S}^T\vec{t}(\vec{u},\dot{\vec{u}},\vec{q},\dot{\vec{q}}) \end{bmatrix} \tag{11}$$

Equation (11) can be imported in a simulation environment, thus computing the values of the accelerations at each step by solving the system, and obtaining the values of velocities and of displacements by integration.

The dynamic model described above was validated by means of experimental tests on real flexible-link mechanisms, by comparing the values of accelerations and elastic deformations experimentally measured with those obtained in simulation. The model was validated using a chain of four flexible bodies [17] and a five-link elastic mechanism, with two-degrees-of-freedom [11].

3 Linearization of the Model

A useful application of the dynamic model described above is the synthesis of controllers for reducing the vibrations of flexible multibody systems. To be able to do that, it is convenient to linearize Eq. (11), so as to bring the model into the state space form. The linearization of the model [10] will be briefly described in the following.

The augmented state-space vector is taken as: $\dot{x}(t) = [\dot{\vec{u}}\ \ \dot{\vec{q}}\ \ \vec{u}\ \ \vec{q}]^T$, so that Eq. (11) becomes:

$$\begin{bmatrix} \vec{M} & \vec{M}\vec{S} & 0 & 0 \\ \vec{S}^T\vec{M} & \vec{S}^T\vec{M}\vec{S} & 0 & 0 \\ 0 & 0 & \vec{I} & 0 \\ 0 & 0 & 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \ddot{\vec{u}} \\ \ddot{\vec{q}} \\ \dot{\vec{u}} \\ \dot{\vec{q}} \end{bmatrix} = \begin{bmatrix} -2\vec{M}_G & -\vec{M}\dot{\vec{S}} & \vec{K} & 0 \\ -2\vec{S}^T\vec{M}_G & \vec{S}^T\vec{M}\dot{\vec{S}} & 0 & 0 \\ \vec{I} & 0 & 0 & 0 \\ 0 & \vec{I} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\vec{u}} \\ \dot{\vec{q}} \\ \vec{u} \\ \vec{q} \end{bmatrix} + \begin{bmatrix} \vec{M} & \vec{I} \\ \vec{S}^T\vec{M} & \vec{S}^T \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{g} \\ \vec{f} \end{bmatrix} \tag{12}$$

In a more compact form:

$$\vec{A}(\vec{x}(t))\dot{\vec{x}}(t) = \vec{B}(\vec{x}(t))\vec{x}(t) + \vec{C}(\vec{x}(t))\vec{v}(t) \tag{13}$$

where the matrices $\vec{A}$, $\vec{B}$ and $\vec{C}$ do not depend on the input vector $\vec{v}$. However, the system in Eq. (12) is still non-linear, because the matrix $\dot{\vec{S}}$ contains the values of the velocities $\dot{\vec{q}}$ of the free coordinates, so quadratic terms appear. If an equilibrium point $\vec{x}_e$, $\vec{v}_e$ is chosen, we can write: $\vec{x}(t) = \vec{x}_e + \Delta\vec{x}(t)$, $\vec{v}(t) = \vec{v}_e + \Delta\vec{v}(t)$, so Eq. (13) becomes:

$$\vec{A}(\vec{x}_e)\Delta\dot{\vec{x}}(t) = \vec{B}(\vec{x}_e + \Delta\vec{x}(t))(\vec{x}_e + \Delta\vec{x}(t)) + \vec{C}(\vec{x}_e + \Delta\vec{x}(t))(\vec{v}_e + \Delta\vec{v}(t)) \tag{14}$$

where it has been taken into account that $\dot{\vec{x}}_e = 0$ for definition of equilibrium point, and the approximation: $\vec{A}(\vec{x}_e + \Delta\vec{x}(t))\Delta\dot{\vec{x}}(t) \cong \vec{A}(\vec{x}_e)\Delta\dot{\vec{x}}(t)$ has been used. The final expression for the system linearized about the equilibrium state is:

$$\vec{A}(\vec{x}_e)\Delta\dot{\vec{x}}(t) = \left[\vec{B}(\vec{x}_e) + \left(\left.\frac{\partial\vec{B}}{\partial\vec{x}}\right|_{\vec{x}=\vec{x}_e} \otimes \vec{x}_e\right) + \left(\left.\frac{\partial\vec{C}}{\partial\vec{x}}\right|_{\vec{x}=\vec{x}_e} \otimes \vec{v}_e\right)\right]\Delta\vec{x}(t) + \vec{C}(\vec{x}_e)\Delta\vec{v}_e(t) \tag{15}$$

where the "$\otimes$" symbol is meant to indicate the inner product of each vector $\left[\frac{\partial B_{i,1}}{\partial x_j} \dots \frac{\partial B_{i,n}}{\partial x_j}\right]_{x=x_e}$, for any i and j, by the vectors $\vec{x}_e$ and $\vec{v}_e$. Once the equilibrium point $\vec{x}_e$ is set, defining the matrices $\vec{A}(\vec{x}_e)$, $\vec{\vec{B}}(\vec{x}_e)$ and $\vec{C}(\vec{x}_e)$ is straightforward, and the matrices $\left(\left.\frac{\partial\vec{B}}{\partial\vec{x}}\right|_{\vec{x}=\vec{x}_e} \otimes \vec{x}_e\right)$ and $\left(\left.\frac{\partial\vec{C}}{\partial\vec{x}}\right|_{\vec{x}=\vec{x}_e} \otimes \vec{v}_e\right)$ can be computed according to their definitions.

The dynamic model described above was then used in order to test in simulation several vibration controllers for flexible multibody systems. For instance, a PID regulator [9], an optimal controller [12], a model predictive controller [1, 2, 23], a delayed reference control [4], as well as hybrid controllers [3, 5, 25] were synthesized and tested, yielding good results. Moreover, the model could be employed in connection with innovative simulation techniques, such as the Hardware-in-the-Loop [32].

4 Extension to the 3D Case

The dynamic model described in this paper was originally intended for planar mechanisms. However, although planar mechanisms are an important category of multibody systems (many industrial machines are based on planar mechanisms), it was convenient to study a dynamic model for 3D mechanisms. As it is known, the extension from the 2D to the 3D case is not straightforward. The 3D dynamical model, based on the considerations above extended to the 3D case, was described in [28, 29]. The extension to the 3D system was done by collocating several reference frames along the kinematic chain, according to the Denavit-Hartenberg rules, and by defining the transformation matrices between any two consecutive frames. So, by using a local to global transformation matrix $\vec{R}_i(\vec{q})$, a block-diagonal rotation matrix $\vec{\vec{T}}_i(\vec{q})$ and an interpolation function matrix $\vec{N}_i(\vec{x}_i, \vec{y}_i, \vec{z}_i)$, one can compute the virtual displacements in the fixed reference frame and the acceleration of a generic point inside the i-th finite element.

As in the 2D case, after defining the kinematics, the dynamic equations of motion can be computed by means of the principle of virtual work, by adding the inertial, elastic and external generalized force terms:

$$dW^{inertia} + dW^{elastic} = -dW^{external} \tag{16}$$

$$\sum_i \int_{v_i} \delta\vec{p}_i^T \ddot{\vec{p}}_i \rho_i d\vec{v} + \sum_i \int_{v_i} \delta\vec{\varepsilon}_i^T \vec{D}_i \vec{\varepsilon}_i d\vec{v} = \sum_i \int_{v_i} \delta\vec{p}_i^T \vec{g}_i \rho_i d\vec{v} + (\delta\vec{u}^T + \delta\vec{r}^T)\vec{f} \tag{17}$$

where ρ_i, $\vec{D}_i$ and ε_i are the mass density, the stress-strain matrix and the strain vector for the i-th element, $\vec{g}$ is the gravity acceleration vector and $\vec{f}$ is the vector of the external forces and torques. As in the 2D case, the nodal elastic virtual displacements $\delta\vec{u}$ and virtual displacements of the ERLS $\delta\vec{r}$ are independent, which yields two set of equilibrium equations, namely those expressing the local and the global equilibrium at the nodes:

$$\vec{M}(\ddot{\vec{r}} + \ddot{\vec{u}}) + 2(\vec{M}_{G1} + \vec{M}_{G2})\dot{\vec{u}} + (\vec{M}_{C1} + 2\vec{M}_{C2} + \vec{M}_{C3})\vec{u} + \vec{K}\vec{u} = (\vec{f}_g + \vec{f}) \tag{18}$$

$$\vec{J}^T\vec{M}(\ddot{\vec{r}} + \ddot{\vec{u}}) + 2\vec{J}^T(\vec{M}_{G1} + \vec{M}_{G2})\dot{\vec{u}} + \vec{J}^T(\vec{M}_{C1} + 2\vec{M}_{C2} + \vec{M}_{C3})\vec{u} = \vec{J}^T(\vec{f}_g + \vec{f}) \tag{19}$$

where $\vec{M}$ is the mass matrix, $\vec{M}_{G1}$ and $\vec{M}_{G2}$ are the Coriolis' terms, $\vec{M}_{C1}$, $\vec{M}_{C2}$ and $\vec{M}_{C3}$ the centrifugal stiffness terms, $\vec{K}$ the stiffness matrix, $\vec{J}$ the Jacobian matrix, and $\vec{f}_g$ the vector of the equivalent nodal loads due to gravity. In order to make the model more realistic, Rayleigh damping was considered and inserted in the model, (α and β coefficients). In matrix form, one can write:

$$\begin{bmatrix} \vec{M} & \vec{M}\vec{J} \\ \vec{J}^T\vec{M} & \vec{J}^T\vec{M}\vec{J} \end{bmatrix} \begin{bmatrix} \ddot{\vec{u}} \\ \ddot{\vec{q}} \end{bmatrix} =$$
$$\begin{bmatrix} -2(\vec{M}_{G1} + \vec{M}_{G2}) - \alpha\vec{M} - \beta\vec{K} & -\vec{M}\dot{\vec{J}} & -(\vec{M}_{C1} + 2\vec{M}_{C2} + \vec{M}_{C3}) - \vec{K} \\ \vec{J}^T(-2(\vec{M}_{G1} + \vec{M}_{G2}) - \alpha\vec{M}) & -\vec{J}^T\vec{M}\dot{\vec{J}} & -\vec{J}^T(\vec{M}_{C1} + 2\vec{M}_{C2} + \vec{M}_{C3}) \end{bmatrix} \begin{bmatrix} \dot{\vec{u}} \\ \dot{\vec{q}} \\ \vec{q} \end{bmatrix} + \begin{bmatrix} \vec{M} & \vec{I} \\ \vec{J}^T\vec{M} & \vec{J}^T \end{bmatrix} \begin{bmatrix} \vec{g} \\ \vec{f} \end{bmatrix} \tag{20}$$

which can be used to run the integration-based simulations.

The 3D model was validated by means of experimental tests, by comparing the measured accelerations and deformations with those provided by simulations [13].

5 From a Nodal to a Modal Approach

In the models described in the foregoing, either 2D or 3D, the ERLS concept has been exploited together with a FEM approach, namely a nodal approach. The latest studies developed an ERLS-based model which could be employed also within a framework based on a modal approach [30]. In this way, one can obtain a more flexible solution based upon a reduced-order system of equations. This is the first work in the literature in which the ERLS concept is used to formulate the dynamics of 3D flexible mechanisms with a component mode synthesis (CMS) approach. The core of the method lies in expressing the nodal displacements $\vec{u}_i$ of the i-th link as functions of a given number of eigenvectors $\vec{U}_i$ and modal coordinates $\vec{q}_i$, namely:

$$\vec{u}_i = \vec{U}_i\vec{q}_i \tag{21}$$

By introducing the local-to-local transformation matrix $\vec{T}_{i+1,i}(\theta)$ between the two reference frames of the ELRS associated to the two consecutive links i and $i+1$:

$$\vec{\hat{u}}_{i+1} = \vec{T}_{i+1,i}\vec{\hat{u}}_i \tag{22}$$

one obtains the following equation:

$$\vec{S}_{i+1}\vec{U}_{i+1}\vec{q}_{i+1} = \vec{T}_{i+1,i}(\theta)\vec{S}_i\vec{U}_i\vec{q}_i \tag{23}$$

which can be rewritten as:

$$\left[-\vec{T}_{i+1,i}(\theta)\vec{S}_i\vec{U}_i \;\; \vec{S}_{i+1}\vec{U}_{i+1}\right]\begin{bmatrix}\vec{q}_i\\ \vec{q}_{i+1}\end{bmatrix} = 0 \tag{24}$$

or:

$$\vec{C}(\vec{\theta})\vec{q} = 0 \tag{25}$$

where $\vec{S}_i$ is the joint displacements selecting matrix, $\vec{C}(\vec{\theta})$ is a band-diagonal matrix, $\vec{q}$ is the modal coordinate vector and $\vec{\theta}$ is the joint parameter one. Starting from this, a quite long dissertation is carried out, in order to get a model. This was then validated by comparing the results of the simulator with those provided by ADAMS-Flex™ software for the same benchmark mechanism (a 3D L-shaped link).

6 Conclusions

In this work, the evolution of a dynamic model for flexible multibody systems, from the original formulation in the 1990s up to the latest developments, was presented. The model is based on an equivalent rigid-link system and originally has been exploited together with a FEM approach for the modeling of planar flexible-link mechanisms. Subsequently, the model has been linearized for control purposes and then it has been extended to the three-dimensional case. In the last years, a modal approach has been developed and the ERLS concept has been applied in order to formulate the dynamics of spatial flexible mechanisms with a component mode synthesis (CMS) technique. In this way, a more flexible solution based upon a reduced-order system of equations can be obtained.

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