Efficient closed-form solution of the kinematics of a tunnel digging machine

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In this work the kinematics of a large size tunnel digging machine is investigated. The closed-loop mechanism is made by 13 links and 13 class 1 couplings, 7 of which are actuated. This kind of machines are commonly used to perform ground drilling for the placement of reinforcement elements during the construction of tunnels. The direct kinematic solution is obtained using three methods: the first two are based on the numerical solution of the closure equation written using the Denavit-Hartenberg convention, while the third is based on the definition and solution in closed form of an equivalent spherical mechanism. The procedures have been tested and implemented with reference to a real commercial tunnel digging machine. The use of the proposed method for the closed–form solution of direct kinematics allows to obtain a major reduction of the computation time in comparison with the standard numerical solution of the closure equation.

1 Introduction

Underground tunnels are excavated using mainly two methods. The first one involves the use of large size tunnel boring machines [1], the other, of more frequent application, is the use of common digging techniques with the use of mechanical excavators and explosives [2]. The excavation resulting form tunnel boring can represent a major problem for the stability of the nearby soil strata and geological structures. The resulting soil displacements must be minimized in order to avoid the possible damage to existing structures, especially in urban areas, where tunnels excavations are usually performed at shallow depths. For this reason the physical properties of the medium to be excavated often requires to perform a reinforcement over the crown of the tunnel by means of artificial structures [3] or by high-pressure injection of concrete, often called jet grouting [4]. This method is particularly popular in Europe, where weak soil structures are reinforced by means of the installation of partially overlapping cylinders of concrete. The idea is to enforce and stabilize the ground material through a support structure ahead and around the excavation. This technique reinforces the work area by creating a supporting system as a reinforced arched shell and allows a fast and safe excavation. Sub-horizontal jet grouting, spiling and pipe roof are other common pre-consolidation methodologies [5].

The jet grouting method improves the strength, stiffness and permeability characteristics of weak soils (e.g. sandy, gravel formations) through the injection of the grouting material, most often a cement-based grout, into boreholes of predetermined shape, size and depth performed on the face

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of the tunnel. This ground modification technique is based on the erosional action of pressurized fluids (i.e. water and grout) that are injected via a special drill tool which is rotated and withdrawn at controlled rates. The purpose is to perform soil-cement columns by means of mixing and partially replacing the surrounding soils with the cementitious grout.

The spiling method, also called umbrella method [6, 7], can be used wherever an extra ground support is required. The purpose of this technique is to maintain a correct arch profile and to create a bridge for unstable rock mass by means of spiling bolts. In this case the pipes are arranged in horizontal or sub-horizontal direction arranged as frustum of conic geometry; the divergence, with respect to the gallery axis, is in the range of $5 \div 10$ degrees. A visual representation of the orientation of the reinforcement elements is shown in Fig. 1. The pipe roof method [8] is based on the installation of a set of parallel steel or concrete pipes around the contour of the tunnel in order to form a ring.

Considering the different working conditions, the typology of pre-consolidation and the spatial constraints inside the galleries, the tunneling machine must be able to reach the target position as fast as possible, assure a high working velocity and allow the optimal consolidation technique with respect to the geological conditions. Many types of drilling

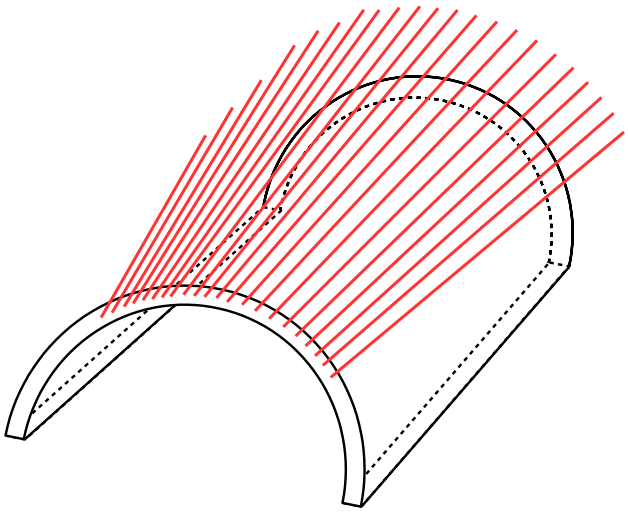

Fig. 1: Placement of reinforcement elements for a crown section: umbrella arch method

machines can be found in working sites [9–11]. Most machines of this kind feature a spatial mechanism to support and move the drilling element, but in many cases the operation of such mechanism is fully demanded to manual control. Therefore the placement of the digging tool and its operation is performed by a skilled operator which sets the position of each individual actuated joint of the machine. In this sense, the analysis of a machine of this type can be useful not only for the development and optimization of existing machines, but also for improving the efficiency of their operation.

The kinematic complexity of most tunnel digging machines may represent a challenge for their mechanical design. Most of them, including the one analyzed in this work, are designed as a closed-loop mechanism with several degrees of freedom, of which only a few of them are actuated. The complexity of computing the direct kinematics for this kind of mechanism must not be underestimated, as for closed-loop mechanisms usually the inverse kinematics is an easier problem to be solved [12]. For this reason the development of these machines is often based on the use of traditional and well established geometries, with refinements performed mainly using CAD programs. When an efficient and fast procedure for computing the direct kinematics is not available, the designer might have to undergo long and tedious trials to test each design in order to compare it to the required specifications. The fulfillment of some of these specifications, such as the extension of the workspace and the space occupation during transport, can be checked efficiently only when a forward kinematic analysis method is made available to the designer. The workspace analysis, in particular, is usually obtained by a discretization of the joint space, followed by the solution to the direct kinematics for each point and the latter verification of the constraints that limit the workspace [13]. A direct kinematic algorithm might be essential also for conducting some sort of design optimization, both from the kinematic point of view [14], and from the structural point of view [15]. The latter, in particular, is gaining wider application in recent years [16] also in industry, given the availability of specialized commercial software tools. For these reasons, the development of more agile tools for the kinematic analysis can hopefully represent an evolution of the common design procedure.

The most general definition of a direct (or forward) kinematics problem is to find the position and orientation of any link of a robot given the geometric structure of the robot and a value of a number of joints position equal to the number of degrees of freedom of the mechanism [17]. This procedure is of paramount importance, since the posture of the robot is usually evaluated from data made available from joint measures. The solution of this kind of problems is quite simple for open-chain robots, since the position of any link can be described by a sequence of independent transformations, with each being defined by each joint position. This operation is usually performed using the Denavit-Hartenberg (D-H) notation [18], using which the end-effector position is uniquely defined by the product of homogeneous transformation matrices of size 4x4.

The simplicity of this approach collapses when a closed-loop manipulator is investigated, i.e. a manipulator in which each joint is connected to two other links using various kinematic pairs. Their possibility to constitute a valid alternative to open chain manipulators for heavy-duty applications [19], together with the complexity of their study, have fostered a vast literature developed since the 60's [20–23]. Often, for this class of mechanisms it is easier to compute the inverse kinematics, i.e. the problem of finding the values of each joint position to achieve the desired pose of the robot. In many cases the direct kinematic problem may also have mul-

tiple solutions: for example, the Gough platform has 40 possible solutions [17]. Such problem can be tackled by defining a closure equation, in which the kinematic constraints are expressed as one nonlinear homogeneous equation for each closed chain of the manipulator. While the definition of the closure equation relies on well-established methods, such as screw theory [24, 25] or, again, D-H notation [26], their numerical solution is often problematic. A numerical solution can be achieved using iterative methods such as the Newton-Raphson or the Newton-Gauss iterative scheme. If the initial guess of the solution is sufficiently accurate, such methods can be quite fast, but in the case that the initial choice is not proper, convergence is easily jeopardized. If a solution is achieved, it is also not guaranteed that the solution is actually feasible, since it can be compatible with another assembly mode. Therefore it might be necessary to repeat the analysis until a desired solution is achieved, or all the possible solutions are achieved. Clearly this procedure does not guarantee any upper bound on the time needed to reach a feasible solution for the direct kinematic problem.

In this paper the general problem addressed above is applied to a specific problem, i.e. the kinematic analysis of a large size tunnel digging machine with closed-loop kinematics. In particular, three methods for solving the direct kinematics problem are proposed, and a comparison of the computational needs for each method is analyzed. The first two methods are based on the solution of a closure equation based on the Denavit-Hartenberg notation [18]. In the first case, a single nonlinear equation is written for the whole kinematic chain and solved numerically. The second option involves the decomposition of the mechanism into three manipulators, two of which are fully actuated and the other is completely passive. Again, the solution to the resulting nonlinear problem is achieved numerically, using the Newton-Raphson method. The third method, which is based on the definition of an equivalent spherical mechanism [27], allows to solve the direct kinematics in a closed-form, therefore without the use of an iterative method. The equivalent spherical mechanism approach is based on the definition of a manipulator designed by translating the directions of the joint axis vectors and the link vectors so that they all intersect at one point. Therefore the equivalent manipulator can be scaled to fit a unit sphere to maintain the correct angular relationship between the joint axis vectors [28]. The advantage brought by this method consists in the availability of trigonometric laws, made available in [27], that allow to express the kinematics as a function of only the joint angles and twist angles of the spatial mechanism.

The development of the solutions, and an analysis of the results obtained with the three methods will be presented in the next sections.

2 Structure of the tunneling machine

The tunneling machine under investigation, which is shown in Fig. 2, is composed of 13 links (including the ground link) and 13 couplings. There are 4 prismatic joints and 9 revolute joints. 7 of the joints are actuated by hydraulic motors.

Fig. 2: The tunnel digging machine, photo courtesy of Casagrande Group

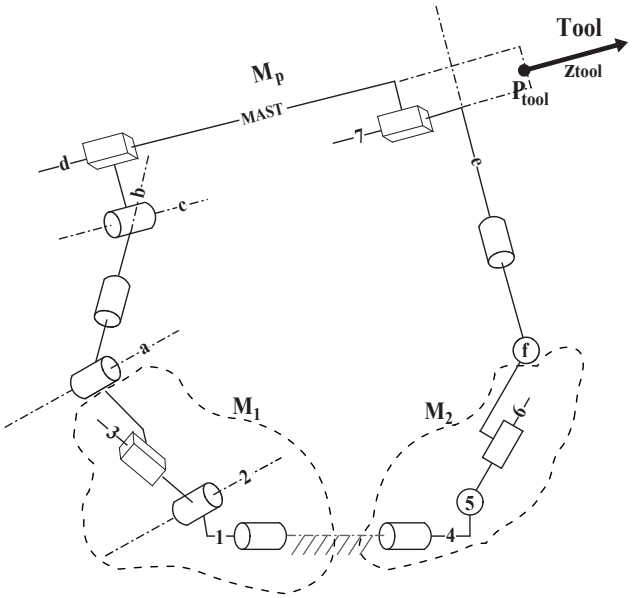

Fig. 3: Kinematic model of the tunneling machine

Figure 3 shows the structure of the kinematic chain of the tunneling machine. The joints are labeled using the letter "c" and an alphanumeric subscript which is a number in the case of an actuated joint, or a letter for a passive joint. The linear joint 7 is used to control the extension of the mast, which is the 24 m long element which holds the perforation tool. To better organize and understand the kinematics of the whole machine, the joints from 1 to 3 are grouped as belonging to the manipulator number 1: $\mathcal{M}_1 = \{c_1, c_2, c_3\}$, while joints from 4 to 6 belong to the manipulator number 2: $\mathcal{M}_2 = \{c_4, c_5, c_6\}$. The remaining joints are grouped as

the manipulator $\mathcal{M}_p = \{c_a, c_b, c_c, c_d, c_7, c_e, c_f\}$. The mast is used to set the orientation and the position of the drilling tool. Precise placement of such element is crucial to perform a suitable hole for the insertion of reinforcement elements such as fiberglass poles or to perform jet grouting.

3 Direct kinematics: numerical solution for the whole mechanism

The direct kinematic problem, i.e. the evaluation of the position and the orientation of the mechanism given the joint variables, can be computed in several ways. Three methods are explored here, leading to three independent ways to achieve a solution. The first two are based on the use of the transformation matrices arising from the kinematic analysis as defined by the Denavit-Hartenberg notation. The first method will be briefly explained in this section.

The most straightforwards solution to the problem of direct kinematics, i.e. the one that involves the smallest amount of preparation, can be achieved with the use of the Denavit-Hartenberg technique [18]. With reference to Fig. 4, the D–H parameters of Table 1 can be defined. The joint variables are indicated as q_i.

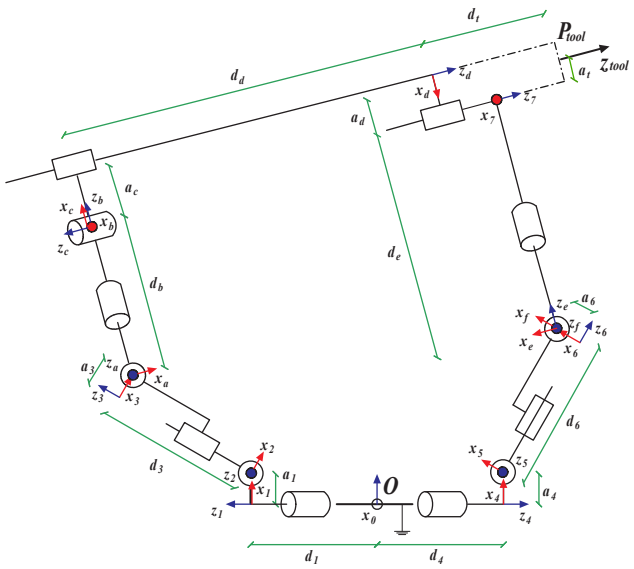

Fig. 4: Denavit-Hartenberg reference system definitions

The closed-loop kinematics of the mechanism allows to write the closure equation as the product of the transformation matrices $\mathbf{T}_i^{i-1}$ as:

$$\mathbf{T}_1^0\mathbf{T}_2^1\mathbf{T}_3^2\ldots\mathbf{T}_6^f\mathbf{T}_6^5\mathbf{T}_5^4\mathbf{T}_4^0 = \mathbf{I} \tag{1}$$

The closure equation can be also written as:

$$\mathbf{T}_0^3\mathbf{T}_c^3 = \mathbf{T}_6^0\mathbf{T}_c^6 \tag{2}$$

i	$\mathbf{T}_i^{i-1}$	α_{i-1}	a_{i-1}	θ_i	S_i
1	$\mathbf{T}_1^0$	$\pi/2$	0	$q_1+\pi/2$	d_1
2	$\mathbf{T}_2^1$	$\pi/2$	a_1	$-q_2$	0
3	$\mathbf{T}_3^2$	$-\pi/2$	0	0	d_3+q_3
a	$\mathbf{T}_a^3$	$\pi/2$	a_3	$-\theta_a$	0
b	$\mathbf{T}_b^a$	$-\pi/2$	0	$\theta_b-\pi/2$	d_b
c	$\mathbf{T}_c^b$	$\pi/2$	0	$\theta_c+\pi/2$	0
d	$\mathbf{T}_d^c$	π	a_c	π	d_d
7	$\mathbf{T}_7^d$	0	a_d	$-\pi/2$	q_7
e	$\mathbf{T}_e^7$	$\pi/2$	0	$\theta_e-\pi/2$	$-d_e$
f	$\mathbf{T}_f^e$	$-\pi/2$	0	$-\theta_f$	0
	$\mathbf{T}_6^f$	$\pi/2$	$-a_6$	0	0
4	$\mathbf{T}_4^0$	$-\pi/2$	0	$q_4-\pi/2$	d_4
5	$\mathbf{T}_5^4$	$-\pi/2$	a_4	q_5	0
6	$\mathbf{T}_6^5$	$\pi/2$	0	0	d_6+q_6

Table 1: Denavit-Hartenberg parameters for the whole mechanism

in which $\mathbf{T}_0^3$ and $\mathbf{T}_6^0$ are known in a direct kinematics problem, being the transformation matrix related to the two fully-actuated manipulators $\mathcal{M}_1$ and $\mathcal{M}_2$. Therefore the contributions due to the passive joints are included in the matrices $\mathbf{T}_c^3$ and $\mathbf{T}_c^6$. Equation (2) includes 6 independent nonlinear equations in the 6 unknown quantities θ_a, θ_b, θ_c, d_d, θ_e, θ_f. The solution to this set of nonlinear equation can be found numerically by use of the Newton-Raphson method [26, 29]. The main drawback of this method is that the numerical solution of the problem must be repeated several times, iterating the procedure using different initial guesses for the solution, until all the eight possible solutions are found. After that, the solutions which are not compliant with the joint limits must be discarded.

4 Direct kinematics: numerical solution for the decomposed mechanism

A more efficient solution can be found by analyzing the manipulator after the decomposition into three submanipulators. According to Fig. 3, the two fully actuated chains $\mathcal{M}_1$ and $\mathcal{M}_2$ can be analyzed after the solution of the kinematics of the rest of the manipulator, which can be identified as $\mathcal{M}_p$. The closure equation for this manipulator can be written in the form:

$$\mathbf{T}_1^5\mathbf{T}_2^1\mathbf{T}_3^2\mathbf{T}_4^3\mathbf{T}_5^4 = \mathbf{I} \tag{3}$$

which can be evaluated using the D-H parameters re-

ported in Table 2, which are based on the notation of Fig. 5.

i	$\mathbf{T}_i^j$	α_{ij}	a_i	θ_i	S_i
1	$\mathbf{T}_1^5$	$\frac{\pi}{2}+q_1+q_4$	a_a	θ_1	d_a
2	$\mathbf{T}_2^1$	$-\frac{\pi}{2}$	0	θ_2	d_b
3	$\mathbf{T}_3^2$	$-\frac{\pi}{2}$	0	θ_3	d_d+q_7
4	$\mathbf{T}_4^3$	$-\frac{\pi}{2}$	0	θ_4	$d_e+a_d-a_c$
5	$\mathbf{T}_5^4$	$\frac{\pi}{2}$	0	θ_5	d_f

Table 2: Denavit-Hartenberg parameters of the manipulator $\mathcal{M}_p$

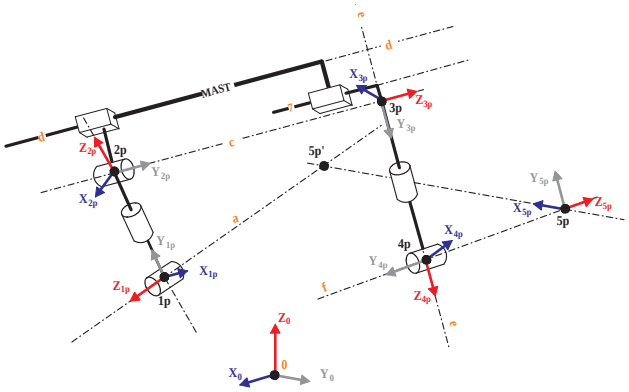

Fig. 5: Reference systems for the manipulator $\mathcal{M}_p$

The evaluation of Eq. (3) requires to compute the unknown values a_a, d_a and d_f. The formulas used to compute them as a function of the joints position of the manipulators $\mathcal{M}_1$ and $\mathcal{M}_2$ are omitted here to comply with the space constraints of the paper. Their graphical representation can be found in Fig. 6. Equation (3) can be computed in the form:

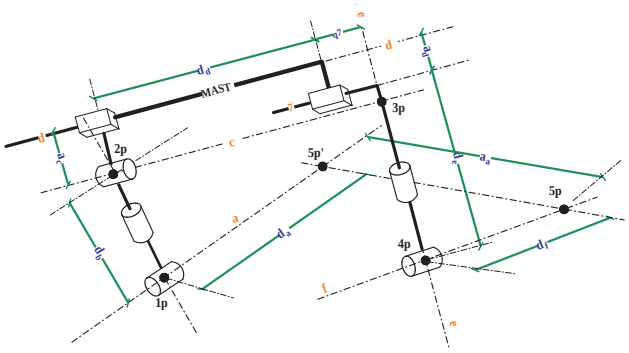

Fig. 6: S_i and d_i measures for the manipulator $\mathcal{M}_p$

$$\begin{bmatrix} N11 & N12 & N13 & N14 \\ N21 & N22 & N23 & N24 \\ N31 & N32 & N33 & N34 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

The matrix equation (4) is equivalent to a system of 12 nonlinear equations in the six unknowns θ_1, θ_2, θ_3, θ_4, θ_5, S_3. In order to find a solution, just 6 equations need to be taken into consideration, i.e.:

$$\begin{cases} f_1(\theta_1,\theta_2,\theta_3,S_3,\theta_4,\theta_5) = N_{11}-1=0 \\ f_2(\theta_1,\theta_2,\theta_3,S_3,\theta_4,\theta_5) = N_{22}-1=0 \\ f_3(\theta_1,\theta_2,\theta_3,S_3,\theta_4,\theta_5) = N_{33}-1=0 \\ f_4(\theta_1,\theta_2,\theta_3,S_3,\theta_4,\theta_5) = N_{14}=0 \\ f_5(\theta_1,\theta_2,\theta_3,S_3,\theta_4,\theta_5) = N_{24}=0 \\ f_6(\theta_1,\theta_2,\theta_3,S_3,\theta_4,\theta_5) = N_{34}=0 \end{cases} \tag{5}$$

Such system can be solved numerically using the Newton-Raphson method. After this, the values just found for the 6 unknown variables must be used to find θ_a, θ_c, θ_e, θ_f and d_d. The relationship between θ_1 and θ_a, as well as the one between θ_5 and θ_f can be understood by using the planes p_a and p_f as shown in Fig. 7. The plane p_a is perpendicular to the axis a, while p_f is normal to the axis f.

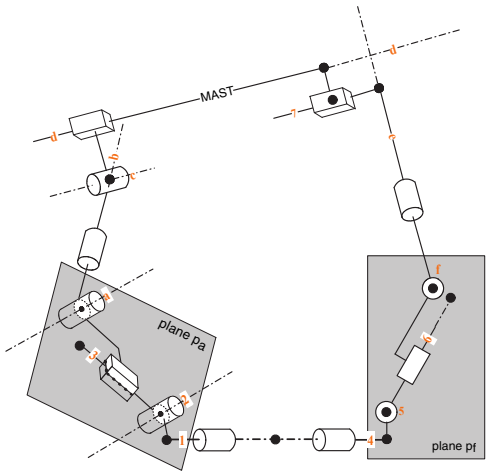

Fig. 7: Planes p_a and p_f

From the analysis of the reference systems represented from a normal direction of the plane p_a, as can be seen in Fig. 8, the following holds:

$$\theta_1 + \frac{\pi}{2} + \theta_a + q_2 - \theta_{par} = 0 \tag{6}$$

from which:

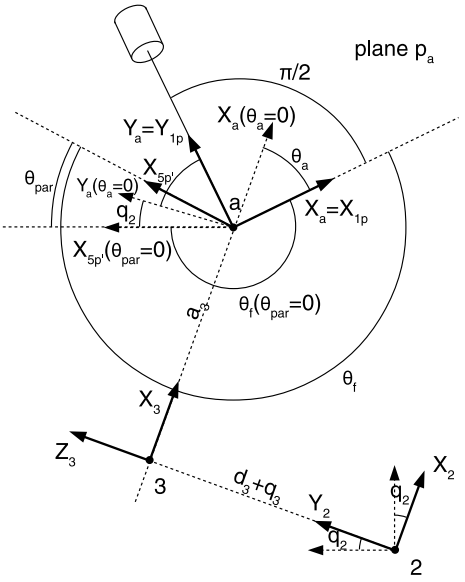

Fig. 8: Reference systems in the plane p_a

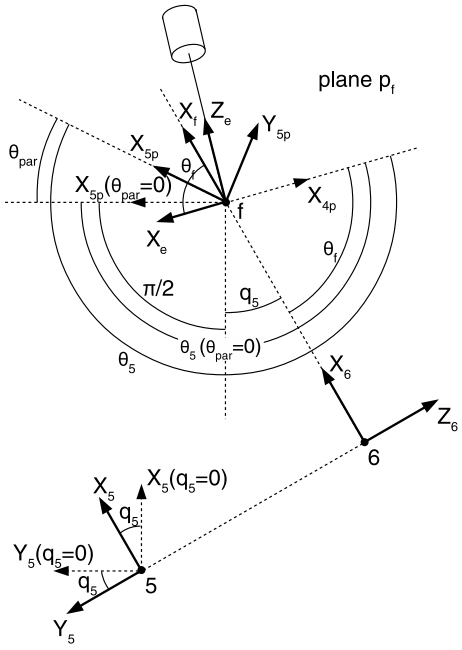

Fig. 9: Reference systems in the plane p_f

$$\theta_a = -\theta_1 - q_2 + \theta_{par} - \frac{\pi}{2} \tag{7}$$

Again, from the projection on the plane p_f, as in Fig. 9:

$$\theta_5 = \theta_{par} + \frac{\pi}{2} + q_5 + \theta_f \tag{8}$$

from which:

$$\theta_f = \theta_5 - q_5 - \theta_{par} - \frac{\pi}{2} \tag{9}$$

The other unknown values can be computed by comparison between the transformation matrices of the whole mechanism and of the decomposed mechanism. By comparing $\mathbf{T}_2^1$ with $\mathbf{T}_a^b$:

$$\theta_b = \theta_2 + \frac{\pi}{2} \tag{10}$$

and from the comparison between $\mathbf{T}_2^3$ and $\mathbf{T}_{c'}^b$:

$$\theta_c = -\theta_3 \tag{11}$$

$$d_d = S_3 - q_7 \tag{12}$$

The last relationship involves the comparison between $\mathbf{T}_4^3$ and $\mathbf{T}_e^{c'}$:

$$\theta_e = -\theta_4 - \frac{\pi}{2} \tag{13}$$

This procedure, as well as the one presented before, is based on the solution of a set of 6 nonlinear equations with 6 unknown values. Despite that the size of the problem is the same, the decomposed solution requires a significantly lower computational effort, given the lower complexity of the analytic expression of each of the six equations. The performance improvement obtained with the second method will be quantified in Sec. 6.1 in terms of average, maximum and minimum time needed to reach a numerical solution.

4.1 Equivalent spherical manipulator

A more efficient solution to the problem can be found by the use of the equivalent spherical mechanism of the manipulator $\mathcal{M}_p$ [27]. It will be shown that the use of this method allows to achieve a dramatic decrease of the computational effort needed to reach a feasible solution to the problem, leading to a closed-form solution that does not require any iteration.

The notation used in this work describes each link of the manipulator by defining the two axes at the extremities of the link through two unitary directional vectors $\mathbf{S}_i$ and $\mathbf{S}_j$. The relative distance between these two vectors is described by the link length a_{ij} and the twist angle α_{ij}. The unity vector $\mathbf{a}_{ij}$ can be defined by the vector product $\mathbf{S}_i \times \mathbf{S}_j = \mathbf{a}_{ij} sin\alpha_{ij}$. Therefore in the case of a rotational joint connecting two

consecutive links, θ_j measures the relative rotation between the two, i.e. the angular distance between the unity vectors $\mathbf{a}_{ij}$ and $\mathbf{a}_{jk}$. For a rotational joint the distance S_j will be a constant.

If a prismatic joint is taken into consideration, the translation of a link jk is performed along an axis parallel to the link ij, leading to a constant angular displacement between ij and jk measured as θ_j. The linear displacement S_j is, in this case, the joint variable. Additionally, a local reference frame can be located on each joint of the manipulator according to this notation: the coordinate system attached to the link ij will have its origin at the intersecting point of vectors $\mathbf{a}_{ij}$ and $\mathbf{S}_i$. The X axis of the reference frame will be parallel to $\mathbf{a}_{ij}$, while the Z axis will be parallel to $\mathbf{S}_i$.

Such notation, that can be applied to define the kinematics of any manipulator, will be used in the rest of the work. As reported by Craig and Duffy in [27], any closed-loop manipulator can be transformed into an equivalent spherical mechanism, for which the kinematics can be solved in a closed form. Such procedure can also be applied to a serial manipulator if a suitable hypothetical closure link is added [27]. Here the procedure will be recalled and applied to the solution of the direct kinematics of the manipulator $\mathcal{M}_p$. The solution of the resultant equivalent mechanism can be obtained in closed-form, achieving therefore a major reduction of the time needed to reach a solution to the direct kinematics problem.

The procedure to be followed to achieve the equivalent spherical manipulator requires to apply a transformation which brings all the unit vectors $\mathbf{S}_i$ of the axes of the couplings to a common intersection point O, which will be the center of an unitary radius sphere. Now the links of the equivalent manipulator can be drawn as arcs of such circumference. The angular displacement between the vector $\mathbf{S}_i$ and the vector $\mathbf{S}_j$ will be α_{ij}. By adding a suitable number of kinematic pairs, the mechanism can be completed. Such procedure, when applied to the manipulator $\mathcal{M}_p$, allows to define the spherical five-bar linkage of Fig. 10: this choice ensures that the original and the equivalent mechanism have both two degrees of freedom.

Given the degree of mobility of the equivalent manipulator, in order to find a solution for the direct kinematics, a system of two independent equations with two unknown values must be solved. Such equations can be originated from a closure equation. For the spherical five-bar linkage the closure equation can be written as:

$$S_1\mathbf{S}_1 + a_{12}\mathbf{a}_{12} + S_2\mathbf{S}_2 + a_{23}\mathbf{a}_{23} + S_3\mathbf{S}_3 + a_{34}\mathbf{a}_{34} + S_4\mathbf{S}_4 + \\ + a_{45}\mathbf{a}_{45} + S_5\mathbf{S}_5 + a_{51}\mathbf{a}_{51} = 0 \tag{14}$$

Equation (14) can be referred to any reference frame and projected on any arbitrary plane, yielding a scalar equation. If Eq. (14) is referred to the first reference frame, the projection along X, Y and Z directions are, respectively:

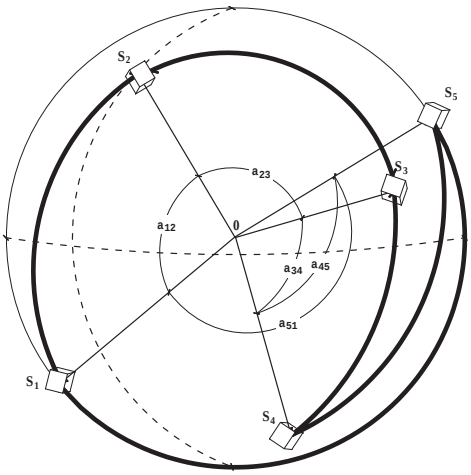

Fig. 10: Equivalent spherical mechanism for the manipulator $\mathcal{M}_p$

$$a_{12} + a_{23}c_2 + S_3\bar{X}_2 + a_{32}W_{34} + S_4X_{32} + a_{45}W_{432} + \\ + S_5X_{432} + a_{51}c_1 = 0$$

$$-S_2s_{12} + a_{23}s_2c_{12} + S_3\bar{Y}_2 - a_{34}U^*_{321} + S_4Y_{32} - a_{45}U^*_{4321} + \\ + S_5Y_{432} - a_{51}s_1 = 0$$

$$S_1 + S_2c_{12} + a_{23}U_{21} + S_3\bar{Z}_2 + a_{34}U_{321} + S_4Z_{32} + a_{45}U_{4321} + \\ + S_5Z_{432} = 0; \tag{15}$$

The above formulation is found by extensive use of the formulas of the cosine direction found in the appendix of reference [27], which are also reported in the appendix of this work, for the reader's reference. The short-hand notations $s_x = sin(x)$ and $c_x = cos(x)$ is used throughout the paper to make the formulas easier to read. In particular, Set 1 has been used to obtain Eq. (15). The following notation is used as well, in order to avoid lengthy expression that involve a long sequence of cosine and sine functions:

$$\bar{X}_j = s_{ij}s_j \tag{16}$$
$$\bar{Y}_j = -(s_{ij}c_{jk} + c_{ij}s_{jk}c_j) \tag{17}$$
$$\bar{Z}_j = c_{ij}c_{jk} - s_{ij}s_{jk}c_j \tag{18}$$
$$X_j = s_{ij}s_j \tag{19}$$
$$Y_j = -(s_{jk}c_{ij} + c_{jk}s_{ij}c_j) \tag{20}$$
$$Z_j = c_{jk}c_{ij} - s_{jk}s_{ij}c_j \tag{21}$$

with the subscript $j = i+1$ and $k = j+1$. Equation (15) is written also by using the following short-hand notation:

$$U_{ji} = s_js_{ij} \tag{22}$$
$$V_{ji} = -(s_ic_j + c_is_jc_{ij}) \tag{23}$$
$$W_{ji} = c_ic_j - s_is_jc_{ij} \tag{24}$$

Other notations are defined in the appendix A or in [27]. It should be highlighted that the scalar product between two vectors is independent from the coordinate system that the two vectors are measured in, so each element of Eq. (15) can be evaluated choosing the most suitable set of equation available in appendix A. The equation (15) takes a general form, that can be written as:

$$\begin{aligned} &_5\left(A_i c_1 + B_i s_1 + D_i\right) + s_5\left(E_i c_1 + F_i s_1 + G_i\right) + \\ &+ \left(H_i c_1 + I_i s_1 + J_i\right) = 0; \quad i = 1,2 \end{aligned} \tag{25}$$

if, for example, the unknown variables are the angles θ_1 and θ_5. By using the trigonometric identities:

$$s_k = \frac{2x_k}{1+x_k^2}; \quad c_k = \frac{1-x_k^2}{1+x_k^2} \tag{26}$$

in which $x_k = tan(\theta_k/2)$, and by multiplication of Eq.(25) by $(1+x_1^2)(1+x_5^2)$ the following is obtained:

$$\begin{aligned} &x_5^2[x_1^2(A_i - D_i - H_i + J_i) + x_1(2I_i - 2B_i) + (-A_i - D_i + H_i + J_i)] \\ &+x_5[x_1^2(-2E_i + 2G_i) + x_1(4F_i) + (2E_i + 2G_i)] \\ &+[x_1^2(-A_i + D_i - H_i + J_i) + x_1(2I_i + 2B_i) + (A_i + D_i + H_i + J_i)] = 0; \\ &\text{for } i = 1,2 \end{aligned} \tag{27}$$

which can also be written, using the Bezout's method [30], as:

$$\begin{aligned} &x_5^2[a_i x_1^2 + b_i x_1 + d_i] + x_5[e_i x_1^2 + f_i x_1 + g_i] + \\ &+[h_i x_1^2 + i_i x_1 + j_i] = 0; \quad \text{for } i = 1,2 \end{aligned} \tag{28}$$

in which the various terms are the following:

$$\begin{array}{lll} a_i = A_i - D_i - H_i + J_i & b_i = 2I_i - 2B_i & d_i = -A_i - D_i + H_i + J_i \\ e_i = 2(G_i - E_i) & f_i = 4F_i & g_i = 2(E_i + G_i) \\ h_i = -A_i + D_i - H_i + J_i & i_i = 2(I_i + B_i) & j_i = A_i + D_i + H_i + J_i \end{array} \tag{29}$$

Equation (28) can be rewritten as:

$$x_5\left(L_1 x_5 + M_1\right) + N_1 = 0; \tag{30}$$

$$x_5\left(L_2 x_5 + M_2\right) + N_2 = 0; \tag{31}$$

in which the coefficients can be evaluated as:

$$\begin{aligned} L_i &= a_i x_1^2 + b_i x_1 + d_i \\ M_i &= e_i x_1^2 + f_i x_1 + g_i \\ N_i &= h_i x_1^2 + i_i x_1 + j_i \end{aligned} \tag{32}$$

The equations (30) and (31) must be linearly dependent in order to find a common solution, which happens for:

$$\begin{vmatrix} L_1 & M_1 \\ L_2 & M_2 \end{vmatrix} \begin{vmatrix} M_1 & N_1 \\ M_2 & N_2 \end{vmatrix} - \begin{vmatrix} L_1 & N_1 \\ L_2 & N_2 \end{vmatrix}^2 = 0 \tag{33}$$

The last equation can be expanded, leading to a eight order polynomial in the variable x_1. In order to evaluate the value of x_5 as a function of the value of x_1, it is sufficient to solve either:

$$x_5 = \frac{-\begin{vmatrix} M_1 & N_1 \\ M_2 & N_2 \end{vmatrix}}{\begin{vmatrix} L_1 & N_1 \\ L_2 & N_2 \end{vmatrix}} \tag{34}$$

or:

$$x_5 = \frac{-\begin{vmatrix} L_1 & N_1 \\ L_2 & N_2 \end{vmatrix}}{\begin{vmatrix} L_1 & M_1 \\ L_2 & M_2 \end{vmatrix}} \tag{35}$$

The eight solutions of the direct kinematics can be obtained using the identities defined for a mechanism belonging to the group 2, according to the notation used in [27].

5 Closed-form direct kinematics for the manipulator $\mathcal{M}_p$

The passive manipulator $\mathcal{M}_p$ can be represented, as specified in the previous section, as a five-bar spherical linkage. A planar representation of this five-bar linkage is provided in Fig. 11, which shows the presence of 4 revolute and one prismatic joint, arranged to form a spherical RRPRR mechanism.

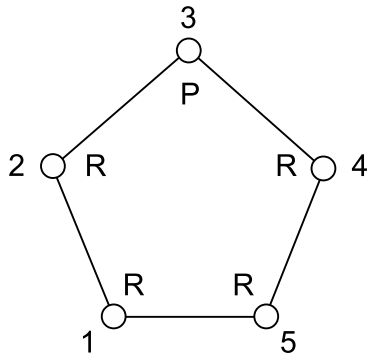

Fig. 11: Planar representation of the spherical five-bar linkage

After the straightforward solution of the kinematics of the serial manipulators $\mathcal{M}_1$ and $\mathcal{M}_2$, the following parameters are known: a_{12}, a_{23}, a_{34}, a_{45}, a_{51}, α_{12}, α_{23}, α_{34}, α_{45}, α_{51}, S_1, S_2, S_4, S_5. The unknown values are: θ_1, θ_2, θ_3, θ_4, θ_5, S_3. The parameters for the representation of the equivalent spherical manipulator according to the Denavit-Hartenberg notation are reported in Table 3.

T_i^j	α_{ji}	a_{ji}	θ_i	S_i
T_1^5	$\pi/2+q_1+q_4$	a_{fa}	θ_1	d_a
T_2^1	$-\pi/2$	0	θ_2	d_b
T_3^2	$-\pi/2$	0	θ_3	d_d+q_7
T_4^3	$-\pi/2$	0	θ_4	$d_e+a_d-a_c$
T_5^4	$\pi/2$	0	θ_5	d_f

Table 3: Denavit-Hartenberg parameters for the equivalent five-bar spherical mechanism

5.1 Projections of the closure equation

The complete procedure developed to solve the direct kinematics of the $\mathcal{M}_p$ manipulator is reported here by making several references to the work [27]. The procedure is shown in detail, since the solution of an RRPRR mechanism is not reported in the mentioned book. The closure equation (14) can be projected on an arbitrary reference system, and for each choice of such reference system, 3 projections, along X,Y ad Z direction, can be computed. Given the fact that the choice of the reference system on which the projection can be made is arbitrary, as stated in the previous section, the choice will be made in order to obtain the simplest possible formulation. The calculation of the scalar equation resulting from the projection is a tedious and complicated task: for this reason the formulas reported here are evaluated using the tables of the direction cosines for a group 2 mechanism reported in the appendix A. Using the 8^{th} set of equation, the projection of Eq. (14) along Z axis is:

$$S_1Z_{23}+a_{12}U_{234}+S_2Z_3+a_{23}U_{34}+S_3c_{34}+S_4+S_5Z_{123}+ +a_{51}U_{1234}=0 \tag{36}$$

and by substituting the values a_{ji} and c_{34} from Table 3, the following is obtained:

$$S_1Z_{23}+S_2Z_3+S_4+S_5Z_{123}+a_{51}U_{1234}=0 \tag{37}$$

For a spherical five-bar linkages the following holds [27]:

$$Z_{23}=\bar{Z}_5 \tag{38}$$
$$Z_3=\bar{Z}_{15} \tag{39}$$
$$Z_{123}=c_{45} \tag{40}$$
$$U_{1234}=s_5s_{45} \tag{41}$$

Since $c_{45}=0$, being $\alpha_{45}=\pi/2$, Eq. (37) can be written as:

$$S_1Z_5+S_2Z_{15}+S_4+a_{51}s_5s_{45}=0 \tag{42}$$

Now, using the correct expression for the terms S_j and making the dependence on angles θ_1 and θ_5 explicit:

$$-c_5s_{51}S_1+S_2(c_1c_5c_{51}-s_1s_5)+S_4+a_{51}s_5=0 \tag{43}$$

A second scalar equation is determined by the projection along the Z axis of the closure Eq. (14), using the set number 10 from the appendix A:

$$S_1c_{12}+a_{12}(0)+S_2(1)+a_{23}(0)+S_3Z_{451}+a_{34}U_{4512}+ +S_4Z_{51}+a_{45}U_{512}+S_5Z_1+a_{51}U_{12}=0 \tag{44}$$

By using the values a_{i-1} from Table 3 and using the relationship $Z_{451}=c_{23}=0$, Eq.(44) can be rewritten as:

$$S_2+S_4Z_{51}+S_5Z_1+a_{51}U_{12}=0 \tag{45}$$

Using the correct expressions for the terms Z_{52} and a_{51} highlights the dependency on the variable θ_1 and θ_5;

$$S_2+S_4(c_1c_5c_{51}-s_1s_5)+S_5c_1s_{51}-a_{51}s_1=0 \tag{46}$$

5.2 Solution of the direct kinematics

The equations (43) and (46) can be used to define a set of nonlinear equations that can be solved in closed-form after some algebraic manipulations that will be described in the following. The procedure allows to write the system of equations (47) in the form of eq. (27):

$$\begin{cases} c_5((S_2c_{51})c_1-s_{51}S_1)+s_5(-S_2s_1+a_{51})+S_4=0 \\ c_5((S_4c_{51})c_1)+s_5((-S_4)s_1)+(S_5s_{51})c_1+(-a_{51})s_1+S_2=0 \end{cases} \tag{47}$$

From a direct comparison between eq. (47) and equations (27,29), the parameters defined in eq. (29) are:

$$
\begin{array}{ll}
a_1 = s_{51}S_1 + c_{51}S_2 + S_4 & a_2 = S_2 + c_{51}S_4 - s_{51}S_5 \\
b_1 = 0 & b_2 = -2a_{51} \\
d_1 = s_{51}S_1 - c_{51}S_2 + S_4 & d_2 = S_2 - c_{51}S_4 + s_{51}S_5 \\
e_1 = 2a_{51} & e_2 = 0 \\
f_1 = -4S_2 & f_2 = -4S_4 \\
g_1 = 2a_5 & g_2 = 0 \\
h_1 = -s_{51}S_1 - c_{51}S_2 + S_4 & h_2 = S_2 - c_{51}S_4 - s_{51}S_5 \\
i_1 = 0 & i_2 = -2a_{51} \\
j_1 = -s_{51}S_1 + c_{51}S_2 + S_4 & j_2 = S_2 + c_{51}S_4 + s_{51}S_5
\end{array} \tag{48}
$$

Using eq. (32), the equation (47) can be written as:

$$
\begin{cases}
x_5(L_1x_5 + M_1) + N_1 = 0 \\
x_5(L_2x_5 + M_2) + N_2 = 0
\end{cases} \tag{49}
$$

in which the terms L_1, M_1, N_1, L_2, M_2 e N_2 are:

$$
\begin{array}{ll}
L_1 = a_1x_1^2 + b_1x_1 + d_1 & L_2 = a_2x_1^2 + b_2x_1 + d_2 \\
M_1 = e_1x_1^2 + f_1x_1 + g_1 & M_2 = e_2x_1^2 + f_2x_1 + g_2 \\
N_1 = h_1x_1^2 + 1_1x_1 + j_1 & N_2 = h_2x_1^2 + 2_2x_1 + j_2
\end{array} \tag{50}
$$

Now, eq. (33) can be rewritten as an 8^{th} order polynomial equation of the unknown x_1:

$$
a_8 + a_7x_1 + a_6x_1^2 + a_5x_1^3 + a_4x_1^4 + a_3x_1^5 + a_2x_1^6 + a_1x_1^7 + a_0x_1^8 = 0 \tag{51}
$$

in which the coefficients from a_0 to a_8 can be evaluated as:

$$
\begin{aligned}
a_0 = {} & 4(-(c_{51}(-d_b^2 + (-a_c + a_d + d_e)^2) + d_b(-d_a + c_{51}d_f)s_{51} + d_ad_fs_{51}^2)^2 - \\
& -a_{fa}^2(d_b^2 - c_{51}^2(-a_c + a_d + d_e)^2 - 2d_bd_fs_{51} + d_f^2s_{51}^2))
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
a_1 = {} & 16a_{fa}((1 + c_{51}^2)d_b(d_b^2 - (-a_c + a_d + d_e)^2) + (2c_{51}d_ad_b^2 - c_{51}^2d_b^2d_f + \\
& +(-2d_b^2 + (-a_c + a_d + d_e)^2)d_f)s_{51} + d_b(d_a^2 - 2c_{51}d_ad_f + d_f^2)s_{51}^2 - \\
& -d_a^2d_fs_{51}^3 + a_{fa}^2(d_b - d_fs_{51}))
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
a_2 = {} & -16(a_{fa}^4 + (d_b^2 - (-a_c + a_d + d_e)^2)^2 + d_b(d_b^2 - (-a_c + a_d + d_e)^2) \\
& (c_{51}d_a - 2d_f + c_{51}^2d_f)s_{51} + (d_a^2(d_b^2 - (-a_c + a_d + d_e)^2) - \\
& -(-1 + c_{51}^2)d_b^2d_f^2)s_{51}^2 - d_ad_bd_f(d_a + c_{51}d_f)s_{51}^3 + \\
& +a_{fa}^2((5 + c_{51}^2)d_b^2 - 2(-a_c + a_d + d_e)^2 + d_b(2c_{51}d_a - 5d_f)s_{51} + d_a^2s_{51}^2))
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
a_3 = {} & -16a_{fa}((-7 + c_{51}^2)d_b(d_b^2 - (-a_c + a_d + d_e)^2) - (2c_{51}d_ad_b^2 + 3c_{51}^2d_b^2d_f + \\
& +(-6d_b^2 + (-a_c + a_d + d_e)^2)d_f)s_{51} + d_b(-3d_a^2 - 2c_{51}d_ad_f + \\
& +d_f^2)s_{51}^2 + d_a^2d_fs_{51}^3 + a_{fa}^2(-7d_b + d_fs_{51}))
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
a_4 = {} & -8(4a_{fa}^4 - (-4 + c_{51}^2)(d_b^2 - (-a_c + a_d + d_e)^2)^2 - (d_a^2(-3d_b^2 + \\
& +4(-a_c + a_d + d_e)^2) + 2c_{51}d_a(-2d_b^2 + (-a_c + a_d + d_e)^2)d_f + \\
& +(4 - 3c_{51}^2)d_b^2d_f^2)s_{51}^2 - d_a^2d_f^2s_{51}^4 + a_{fa}^2((27 - 4c_{51}^2)d_b^2 + \\
& +(-8 + c_{51}^2)(-a_c + a_d + d_e)^2 + (4d_a^2 - d_f^2)s_{51}^2))
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
a_5 = {} & 16a_{fa}((7 - c_{51}^2)d_b(d_b^2 - (-a_c + a_d + d_e)^2) - (2c_{51}d_ad_b^2 + 3c_{51}^2d_b^2d_f + \\
& +(-6d_b^2 + (-a_c + a_d + d_e)^2)d_f)s_{51} + d_b(3d_a^2 + 2c_{51}d_ad_f - d_f^2)s_{51}^2 + \\
& +d_a^2d_fs_{51}^3 + a_{fa}^2(7d_b + d_fs_{51}))
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
a_6 = {} & -16(a_{fa}^4 + (d_b^2 - (-a_c + a_d + d_e)^2)^2 - d_b(d_b^2 - (-a_c + a_d + d_e)^2) \\
& (c_{51}d_a - 2d_f + c_{51}^2d_f)s_{51} + (d_a^2(d_b^2 - (-a_c + a_d + d_e)^2) - \\
& -(-1 + c_{51}^2)d_b^2d_f^2)s_{51}^2 + d_ad_bd_f(d_a + c_{51}d_f)s_{51}^3 + a_{fa}^2((5 + c_{51}^2)d_b^2 - \\
& -2(-a_c + a_d + d_e)^2 + d_b(-2c_{51}d_a + 5d_f)s_{51} + d_a^2s_{51}^2))
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
a_7 = {} & 16a_{fa}((1 + c_{51}^2)d_b(d_b^2 - (-a_c + a_d + d_e)^2) + (-2c_{51}d_ad_b^2 + \\
& +c_{51}^2d_b^2d_f + (2d_b^2 - (-a_c + a_d + d_e)^2)d_f)s_{51} + d_b(d_a^2 - \\
& -2c_{51}d_ad_f + d_f^2)s_{51}^2 + d_a^2d_fs_{51}^3 + a_{fa}^2(d_b + d_fs_{51}))
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
a_8 = {} & -4((c_{51}(-d_b^2 + (-a_c + a_d + d_e)^2) + d_b(d_a - c_{51}d_f)s_{51} + d_ad_fs_{51}^2)^2 + \\
& +a_{fa}^2(d_b^2 - c_{51}^2(-a_c + a_d + d_e)^2 + 2d_bd_fs_{51} + d_f^2s_{51}^2))
\end{aligned} \tag{60}
$$

It has been observed that, when using the kinematic parameters of the machine under investigation, Eq.(51) has four complex and four real solutions. Once these solutions are found, x_5 can be evaluated through eq. (34), whose explicit form is:

$$
\begin{aligned}
x_5 = {} & (-2x_1a_{fa}x_1d_b - 2x_1a_{fa}x_1c_{51}x_1(-a_c + a_d + d_e) - 2x_1a_{fa}x_1d_fx_1s_{51} + \\
& +4x_1a_{fa}^2x_1x_1 + 4x_1d_b^2x_1x_1 - 4x_1(-a_c + a_d + d_e)^2x_1x_1 + 4x_1d_ax_1 \\
& (-a_c + a_d + d_e)x_1s_{51}x_1x_1 + 4x_1d_bx_1d_fx_1s_{51}x_1x_1 - 12x_1a_{fa}x_1d_bx_1x_1^2 + \\
& +4x_1a_{fa}^2x_1x_1^3 + 4x_1d_b^2x_1x_1^3 - 4x_1(-a_c + a_d + d_e)^2x_1x_1^3 + \\
& +4x_1d_ax_1(-a_c + a_d + d_e)x_1s_{51}x_1x_1^3 - 4x_1d_bx_1d_fx_1s_{51}x_1x_1^3 - \\
& -2x_1a_{fa}x_1d_bx_1x_1^4 + 2x_1a_{fa}x_1c_{51}x_1(-a_c + a_d + d_e)x_1x_1^4 + \\
& +2x_1a_{fa}x_1d_fx_1s_{51}x_1x_1^4)/(-2x_1c_{51}x_1d_b^2 + 2x_1c_{51}x_1(-a_c + a_d + d_e)^2 \\
& +2x_1d_ax_1d_bx_1s_{51} - 2x_1c_{51}x_1d_bx_1d_fx_1s_{51} + 2x_1d_ax_1d_fx_1s_{51}^2 \\
& +4x_1a_{fa}x_1c_{51}x_1d_bx_1x_1 - 4x_1a_{fa}x_1d_ax_1s_{51}x_1x_1 + 4x_1d_ax_1d_bx_1s_{51}x_1x_1^2 + \\
& +4x_1c_{51}x_1d_bx_1d_fx_1s_{51}x_1x_1^2 - 4x_1a_{fa}x_1c_{51}x_1d_bx_1x_1^3 - \\
& -4x_1a_{fa}x_1d_ax_1s_{51}x_1x_1^3 + 2x_1c_{51}x_1d_b^2x_1x_1^4 - \\
& -2x_1c_{51}x_1(-a_c + a_d + d_e)^2x_1x_1^4 + 2x_1d_ax_1d_bx_1s_{51}x_1x_1^4 - \\
& -2x_1c_{51}x_1d_bx_1d_fx_1s_{51}x_1x_1^4 - 2x_1d_ax_1d_fx_1s_{51}^2x_1x_1^4);
\end{aligned} \tag{61}
$$

Now, the angular displacements θ_1 and θ_5 can be evaluated using:

$$
\theta_i = 2atan(x_i); \quad i = 1,5 \tag{62}
$$

From the formulas of the direction cosines $Z_{15} = Z_3$, the cosine of θ_3 can be evaluated as:

$$
c_3 = s_1s_5 - c_1c_5c_{15} \tag{63}
$$

From c_3, s_3 can be evaluated as:

$$
s_3 = \sqrt{1 - c_3^2} \tag{64}
$$

Each of the four values of c_3 allows to find two values for s_3, defining thus all the eight solutions of the direct kinematic problem. Now, using the projection along the X axis

of the closure equation (14) using the set 6 from appendix A, the following is obtained:

$$S_2 s_{12} s_1 + S_3 X_{21} + S_4 X_5 + a_{51} = 0 \tag{65}$$

which, by using $s_{12} = -1$, being $\alpha_{12} = -\pi/2$, can be written as:

$$-S_2 s_1 + S_3 c_1 s_2 + S_4 s_5 + a_{51} = 0 \tag{66}$$

Now, being from the fundamental formulas $Y_{234} = s_{51} c_5$ and since the equivalence $Z_{32} = Z_5$ can be written as $-s_2 s_3 = c_5 s_{51}$, the expression $s_2 = -c_5 c_{51}/s_3$ can be used in the last equation leading to:

$$S_3 = -\frac{S_2 s_1 - S_4 s_5 - a_{51}}{c_1 c_5 s_{51}} s_3 \tag{67}$$

From the fundamental relationship $X_{32} = X_{51}$, c_2 can be evaluated as:

$$c_2 = -\frac{c_5 c_{51} s_1 + c_1 s_5}{s_3} \tag{68}$$

From $Z_{32} = Z_5$, s_2 can be evaluated as:

$$s_2 = -\frac{c_5 s_{51}}{s_3} \tag{69}$$

From $Y_{123} = s_{45} c_4$, c_4 is:

$$c_4 = -c_2 c_{51} - s_1 s_2 s_{51} \tag{70}$$

Again, from $X_{123} = s_{45} s_4$, s_4 can be evaluated as:

$$s_4 = -c_1 s_3 s_{51} + c_3(-c_{51} s_2 + c_2 s_1 s_{51}) \tag{71}$$

At last, the angular displacements θ_2, θ_3 and θ_4 can be simply evaluated using:

$$\theta_i = atan\left(\frac{s_i}{c_i}\right); \quad i = 2,3,4 \tag{72}$$

The formulation above is not valid in the case in which $q_1 = q_4$, which happens when the manipulators $\mathcal{M}_1$ and $\mathcal{M}_2$ are lying on the same plane. In this situation $\alpha_{51} = 0$ and therefore eq. (67) is singular. A different relationship from eq. (67) can be established using the projection on the Z axis of the closure equation, using Set 1, i.e.:

$$S_2 c_{12} + S_3 \bar{Z}_2 + S_4 Z_{32} + a_{51} = 0 \tag{73}$$

whose explicit form is:

$$-c_2 S_3 + s_2 s_3 S_4 = 0 \tag{74}$$

But $s_2 s_3 = 0$ from the secondary relationship $Y_{51} = -X^*_{32}$, allowing to write eq.(74) as:

$$-c_2 S_3 = 0 \tag{75}$$

from which $c_2 = 0$. Such value can be substituted in the fundamental relationship $X_{512} = s_{34} s_3$ together with the kinematic parameters of the manipulator, leading to:

$$s_3 = 0 \tag{76}$$

Again, $c_4 = 0$ can be found from $Z_{21} = Z_4$. Now the projection of the closure equation on the X axis, evaluated using the Set 1 is:

$$S_3 \bar{X}_2 + S_4 X_{32} + a_{51} c_1 = 0 \tag{77}$$

which implies that $-s_2 S_3 - c_2 s_3 S_4 + a_{51} c_1 = 0$. Since also $c_2 = 0$, the last equation is equivalent to:

$$s_2 S_3 = a_{51} c_1 \tag{78}$$

Now, since s_2 is equal to ± 1, being $c_2 = 0$, S_3 can be evaluated as:

$$S_3 = \pm a_{51} c_1 \tag{79}$$

At last, the projection of the closure equation using set 7 along the X axis is:

$$S_1 X_{234} S_2 X_{34} + S_3 X_4 + a_{51} c_5 = 0 \tag{80}$$

Using the relationships $X_{234} = s_{51}s_5 = 0$, which is valid being $s_{51} = 0$, rewritten as $-s_4S_3 + a_{51}c_5 = 0$, s_4 can be evaluated as:

$$s_4 = \frac{a_{51}c_5}{S_3} \tag{81}$$

The formulas presented in this section allow to determine all the unknown values θ_i with $i = 1 \dots 5$ and S_3, and therefore by using these the closed-form solution of the direct kinematic problem can be achieved.

6 Results

An example of the solutions that can be found using the three methods described above for the direct kinematic problem is reported here. The input parameters are the joint positions:

$$q_1 = 0.5061\,rad \quad q_2 = 0.3491\,rad \quad q_3 = 1\,m$$
$$q_4 = -0.3491\,rad \quad q_5 = 1.0472\,m \quad q_6 = 1.5\,m$$
$$q_7 = 1\,m$$

The eight solutions of the direct kinematic problem, tested by comparing the results obtained with the three methods explained above, are reported in Table 4 and represented graphically in Fig. 12, 13, 14 and 15. The comparison between the outcome of the three methods presented in this work verifies the consistency of their results.

solution	θ_a [deg]	θ_b [deg]	θ_c [deg]	θ_e [deg]	θ_f [deg]	d_d [m]
a	237.58	3.29	8.79	358.68	222.77	-13.54
b	237.58	183.29	351.21	178.68	222.77	11.54
c	234.84	3.71	188.69	181.33	45.54	-13.53
d	234.84	183.71	171.31	1.33	45.54	11.53
e	57.62	176.29	8.79	181.74	42.80	-13.54
f	57.62	356.29	351.21	1.74	42.80	11.54
g	60.36	176.71	188.88	358.25	220.03	-13.53
h	60.36	356.71	171.12	178.25	220.03	11.53

Table 4: Solutions to the direct kinematic problem

6.1 Evaluation of the computational effort

The computational requirements of the developed algorithms have been tested extensively to evaluate the actual time needed to find the solutions to the direct kinematics problem. For this purpose, 1000 random values for the joint positions of manipulator $\mathcal{M}_1$ and $\mathcal{M}_2$ have been generated, and three different algorithms have been applied to them: the Newton-Raphson method applied to the closure equation for the whole kinematic chain, as in eq. (1), the Newton-Raphson applied to each manipulator separately, as in eq. (2), and the equivalent spherical mechanism approach. The tests have been performed on a low-power PC with just 512 MB of RAM, in order to use a calculator with computational resources similar to those of an industrial portable PC.

	Newton-Rapshon (1) [s]	Newton-Raphson (2) [s]	Spherical equivalent [s]
1000 solutions	5635	688.852	8.76
mean time	5.635	0.689	0.0088
maximum time	24.266	2.063	0.0940
minimum time	2.8910	0.3280	0.0001

Table 5: Overall, average, minimum and maximum time needed to solve the direct kinematics for 1000 random configurations

The results of this simple test are shown in Table 5. If the mean time is taken as the performance index, the algorithm based on the equivalent spherical mechanism is roughly 78 times faster than the Newton-Raphson method applied to the separate solution of the three kinematic chains, and 643 times faster in comparison with the solution for the whole manipulator. Moreover, the spherical equivalent method allows to achieve a solution in time that is always less than 0.1 s, while the two other solution methods can require up to 24 s and 2 s.

The reduced time needed to compute the direct kinematics can be exploited to perform tasks such as the analysis of the workspace, which can be useful for the structural optimization of the machine. The workspace analysis is usually performed by repeating the direct kinematics analysis for a wide range of configurations of the joints positions, in order to cover a wide range of reachable poses of the robot. Such procedure can require easily hundreds of thousands of successive kinematic analysis: in this case the use of a faster algorithm can solve the task in a few minutes instead of several hours.

7 Conclusions

In this work the problem of solving the direct kinematic problem for a tunnel digging machine has been investigated. The machine under investigation is a parallel kinematics robot with 13 links and 13 joints, of which just 7 are actuated. The solution to the forward kinematics is performed using two traditional techniques, i.e. the numerical solution, obtained with the Newton-Raphson method, of a closure equation defined using the Denavit-Hartenberg notation. Such algorithms are tested both for the whole kinematic chain and for a structural decomposition between actuated sub-mechanisms and a passive mechanism. A third method, based on the use of an equivalent spherical mechanism is developed as well. Such approach, allowing to write the solution to the direct kinematic problem in a closed form, bears a relevant reduction of the time needed to perform the forward kinematic solution. The consistency between the solutions brought by the three methods is verified, and the computational requirements for the three methods are compared as well.

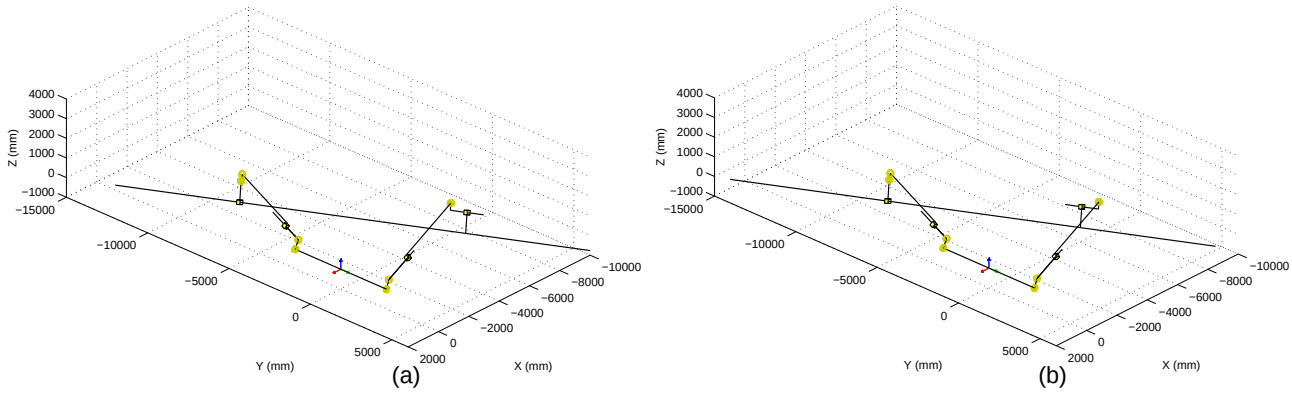

Fig. 12: Solutions (a) and (b) to the direct kinematic problem

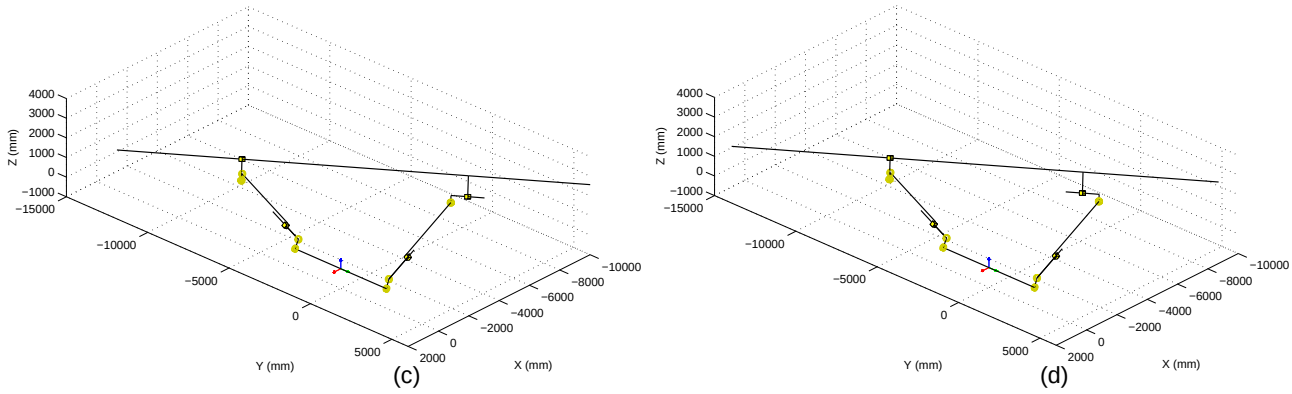

Fig. 13: Solutions (c) and (d) to the direct kinematic problem

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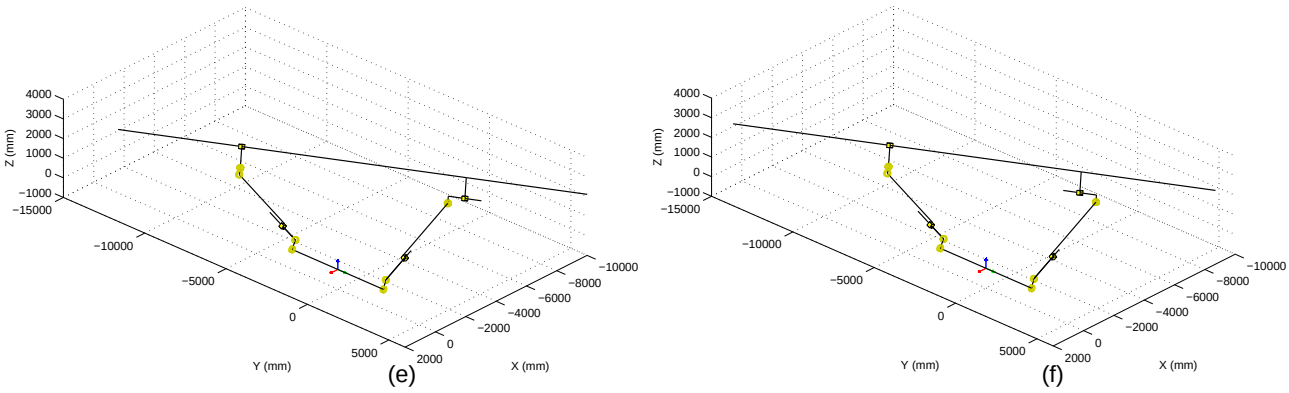

Fig. 14: Solutions (e) and (f) to the direct kinematic problem

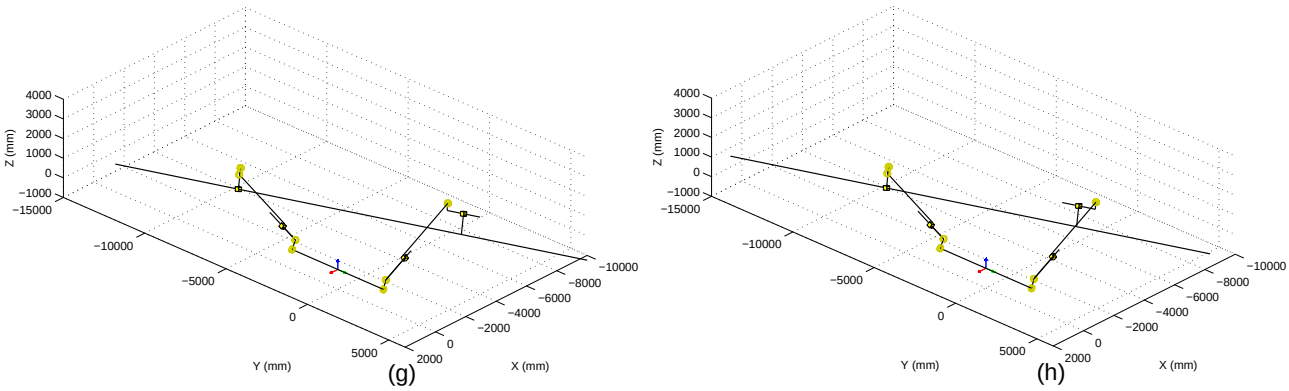

Fig. 15: Solutions (g) and (h) to the direct kinematic problem

[9] Casagrande s.p.a. `http://www.casagrandegroup.com`.

[10] Enteco s.r.l. `http://www.entecodrill.com`.

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Appendix A: Equations for a Spherical Pentagon

Fundamental Formulas:

$$\begin{array}{llll} X_{123}=s_{45}s_4 & Y_{123}=s_{45}c_4 & Z_{123}=c_{45} \\ X_{234}=s_{51}s_5 & Y_{234}=s_{51}c_5 & Z_{234}=c_{51} \\ X_{345}=s_{12}s_1 & Y_{345}=s_{12}c_1 & Z_{345}=c_{12} \\ X_{451}=s_{23}s_2 & Y_{451}=s_{23}c_2 & Z_{451}=c_{23} \\ X_{512}=s_{34}s_3 & Y_{512}=s_{34}c_3 & Z_{512}=c_{34} \end{array}$$

$$\begin{array}{llll} X_{321}=s_{45}s_5 & Y_{321}=s_{45}c_5 & Z_{321}=c_{45} \\ X_{432}=s_{51}s_1 & Y_{432}=s_{51}c_1 & Z_{432}=c_{51} \\ X_{543}=s_{12}s_2 & Y_{543}=s_{12}c_2 & Z_{543}=c_{12} \\ X_{154}=s_{23}s_3 & Y_{154}=s_{23}c_3 & Z_{154}=c_{23} \\ X_{215}=s_{34}s_4 & Y_{215}=s_{34}c_4 & Z_{215}=c_{34} \end{array}$$

Secondary Formulas:

$$\begin{array}{llll} Set1 & X_{123}=\bar{X}_4 & -X^*_{123}=\bar{Y}_4 & Z_{12}=\bar{Z}_4 \\ & X_{234}=\bar{X}_5 & -X^*_{234}=\bar{Y}_5 & Z_{23}=\bar{Z}_5 \\ & X_{345}=\bar{X}_1 & -X^*_{345}=\bar{Y}_1 & Z_{34}=\bar{Z}_1 \\ & X_{451}=\bar{X}_2 & -X^*_{451}=\bar{Y}_2 & Z_{45}=\bar{Z}_2 \\ & X_{512}=X_3 & -X^*_{512}=Y_3 & Z_{51}=Z_3 \end{array}$$

$$\begin{array}{llll} & X_{321}=X_5 & -X^*_{321}=Y_5 & Z_{32}=Z_5 \\ & X_{432}=X_1 & -X^*_{432}=Y_1 & Z_{43}=Z_1 \\ & X_{543}=X_2 & -X^*_{543}=Y_2 & Z_{54}=Z_2 \\ & X_{154}=X_3 & -X^*_{154}=Y_3 & Z_{15}=Z_3 \\ & X_{215}=X_4 & -X^*_{215}=Y_4 & Z_{21}=Z_4 \end{array}$$

$$\begin{array}{llll} Set2 & X_{12}=X_{43} & Y^*_{12}=-X^*_{43} & Z_{12}=\bar{Z}_4 \\ & X_{23}=X_{54} & Y^*_{23}=-X^*_{54} & Z_{23}=\bar{Z}_5 \\ & X_{34}=X_{15} & Y^*_{34}=-X^*_{15} & Z_{34}=\bar{Z}_1 \\ & X_{45}=X_{21} & Y^*_{45}=-X^*_{21} & Z_{45}=\bar{Z}_2 \\ & X_{51}=X_{32} & Y^*_{51}=-X^*_{32} & Z_{51}=\bar{Z}_3 \end{array}$$

$$\begin{array}{llll} & X_{32}=X_{51} & Y^*_{32}=-X^*_{51} & Z_{32}=Z_5 \\ & X_{43}=X_{12} & Y^*_{43}=-X^*_{12} & Z_{43}=Z_1 \\ & X_{54}=X_{23} & Y^*_{54}=-X^*_{23} & Z_{54}=Z_2 \\ & X_{15}=X_{34} & Y^*_{15}=-X^*_{34} & Z_{15}=Z_3 \\ & X_{21}=X_{45} & Y^*_{21}=-X^*_{45} & Z_{21}=Z_4 \end{array}$$

Equations for a spherical pentagon

Fundamental formulas:

$$\begin{array}{lll} U_{1234}=s_{45}s_5 & V_{1234}=c_{45}s_5 & W_{1234}=c_5 \\ U_{2345}=s_{51}s_1 & V_{2345}=c_{51}s_1 & W_{2345}=c_1 \\ U_{3451}=s_{12}s_2 & V_{3451}=c_{12}s_2 & W_{3451}=c_2 \\ U_{4512}=s_{23}s_3 & V_{4512}=c_{23}s_3 & W_{4512}=c_3 \\ U_{5123}=s_{34}s_4 & V_{5123}=c_{34}s_4 & W_{5123}=c_4 \end{array}$$

$$\begin{array}{lll} U_{4321}=s_{51}s_5 & V_{4321}=c_{51}s_5 & W_{4321}=c_5 \\ U_{3215}=s_{45}s_4 & V_{3215}=c_{45}s_4 & W_{3215}=c_4 \\ U_{2154}=s_{34}s_3 & V_{2154}=c_{34}s_3 & W_{2154}=c_3 \\ U_{1543}=s_{23}s_2 & V_{1543}=c_{23}s_2 & W_{1543}=c_2 \\ U_{5432}=s_{12}s_1 & V_{5432}=c_{12}s_1 & W_{5432}=c_1 \end{array}$$

Direction Cosines - Polar Pentagon:

$$\begin{array}{lll} Set1 & \mathbf{S}_1\ (0,0,1) & \mathbf{a}_{12}\ (1,0,0) \\ & \mathbf{S}_2\ (0,-s_{12},c_{12},) & \mathbf{a}_{23}\ (c_2,s_2c_{12},U_{21}) \\ & \mathbf{S}_3\ (\bar{X}_2,\bar{Y}_2,Z_2) & \mathbf{a}_{34}\ (W_{32},-U^*_{321},U_{321}) \\ & \mathbf{S}_4\ (X_{32},Y_{32},Z_{32}) & \mathbf{a}_{45}\ (W_{432},-U^*_{4321},U_{4321}) \\ & \mathbf{S}_5\ (X_{432},Y_{432},Z_{432}) & \mathbf{a}_{51}\ (c_1,-s_1,0) \end{array}$$

$Set2$ $\mathbf{S}_2\ (0,0,1)$ $\mathbf{a}_{23}\ (1,0,0)$
$\mathbf{S}_3\ (0, -s_{23}, c_{23},)$ $\mathbf{a}_{34}\ (c_3, s_3c_{23}, U_{32})$
$\mathbf{S}_4\ (X_3, Y_3, Z_3)$ $\mathbf{a}_{45}\ (W_{43}, -U^*_{432}, U_{432})$
$\mathbf{S}_5\ (X_{43}, Y_{43}, Z_{43})$ $\mathbf{a}_{51}\ (W_{543}, -U^*_{5432}, U_{5432})$
$\mathbf{S}_1\ (X_{543}, Y_{543}, Z_{543})$ $\mathbf{a}_{12}\ (c_2, -s_2, 0)$

$Set3$ $\mathbf{S}_3\ (0,0,1)$ $\mathbf{a}_{34}\ (1,0,0)$
$\mathbf{S}_4\ (0, -s_{34}, c_{34},)$ $\mathbf{a}_{45}\ (c_4, s_4c_{34}, U_{43})$
$\mathbf{S}_5\ (X_4, Y_4, Z_4)$ $\mathbf{a}_{51}\ (W_{54}, -U^*_{543}, U_{543})$
$\mathbf{S}_1\ (X_{54}, Y_{54}, Z_{54})$ $\mathbf{a}_{12}\ (W_{154}, -U^*_{1543}, U_{1543})$
$\mathbf{S}_2\ (X_{154}, Y_{154}, Z_{154})$ $\mathbf{a}_{23}\ (c_3, -s_3, 0)$

$Set4$ $\mathbf{S}_4\ (0,0,1)$ $\mathbf{a}_{45}\ (1,0,0)$
$\mathbf{S}_5\ (0, -s_{45}, c_{45},)$ $\mathbf{a}_{51}\ (c_5, s_5c_{45}, U_{54})$
$\mathbf{S}_1\ (X_5, Y_5, Z_5)$ $\mathbf{a}_{12}\ (W_{15}, -U^*_{154}, U_{154})$
$\mathbf{S}_2\ (X_{15}, Y_{15}, Z_{15})$ $\mathbf{a}_{23}\ (W_{215}, -U^*_{2154}, U_{2154})$
$\mathbf{S}_3\ (X_{215}, Y_{215}, Z_{215})$ $\mathbf{a}_{34}\ (c_4, -s_4, 0)$

$Set5$ $\mathbf{S}_5\ (0,0,1)$ $\mathbf{a}_{51}\ (1,0,0)$
$\mathbf{S}_1\ (0, -s_{51}, c_{51},)$ $\mathbf{a}_{12}\ (c_1, s_1c_{51}, U_{15})$
$\mathbf{S}_2\ (X_1, Y_1, Z_1)$ $\mathbf{a}_{23}\ (W_{21}, -U^*_{215}, U_{215})$
$\mathbf{S}_3\ (X_{21}, Y_{21}, Z_{21})$ $\mathbf{a}_{34}\ (W_{321}, -U^*_{3215}, U_{3215})$
$\mathbf{S}_4\ (X_{321}, Y_{321}, Z_{321})$ $\mathbf{a}_{45}\ (c_5, -s_5, 0)$

$Set6$ $\mathbf{S}_1\ (0,0,1)$ $\mathbf{a}_{51}\ (1,0,0)$
$\mathbf{S}_5\ (0, s_{51}, c_{51},)$ $\mathbf{a}_{45}\ (c_5, -s_5c_{51}, U_{51})$
$\mathbf{S}_4\ (X_5, -Y_5, Z_5)$ $\mathbf{a}_{34}\ (W_{45}, U^*_{451}, U_{451})$
$\mathbf{S}_3\ (X_{45}, -Y_{45}, Z_{45})$ $\mathbf{a}_{23}\ (W_{345}, U^*_{3451}, U_{3451})$
$\mathbf{S}_2\ (X_{345}, -Y_{345}, Z_{345})$ $\mathbf{a}_{12}\ (c_1, s_1, 0)$

$Set7$ $\mathbf{S}_5\ (0,0,1)$ $\mathbf{a}_{45}\ (1,0,0)$
$\mathbf{S}_4\ (0, s_{45}, c_{45},)$ $\mathbf{a}_{34}\ (c_4, -s_4c_{45}, U_{45})$
$\mathbf{S}_3\ (X_4, -Y_4, Z_4)$ $\mathbf{a}_{23}\ (W_{34}, U^*_{345}, U_{345})$
$\mathbf{S}_2\ (X_{34}, -Y_{34}, Z_{34})$ $\mathbf{a}_{12}\ (W_{234}, U^*_{2345}, U_{2345})$
$\mathbf{S}_1\ (X_{234}, -Y_{234}, Z_{234})$ $\mathbf{a}_{51}\ (c_5, s_5, 0)$

$Set8$ $\mathbf{S}_4\ (0,0,1)$ $\mathbf{a}_{34}\ (1,0,0)$
$\mathbf{S}_3\ (0, s_{34}, c_{34},)$ $\mathbf{a}_{23}\ (c_3, -s_3c_{34}, U_{34})$
$\mathbf{S}_2\ (X_3, -Y_3, Z_3)$ $\mathbf{a}_{12}\ (W_{23}, U^*_{234}, U_{234})$
$\mathbf{S}_1\ (X_{23}, -Y_{23}, Z_{23})$ $\mathbf{a}_{51}\ (W_{123}, U^*_{1234}, U_{1234})$
$\mathbf{S}_5\ (X_{123}, -Y_{123}, Z_{123})$ $\mathbf{a}_{45}\ (c_4, s_4, 0)$

$Set9$ $\mathbf{S}_3\ (0,0,1)$ $\mathbf{a}_{23}\ (1,0,0)$
$\mathbf{S}_2\ (0, s_{23}, c_{23},)$ $\mathbf{a}_{12}\ (c_2, -s_2c_{23}, U_{23})$
$\mathbf{S}_1\ (X_2, -Y_2, Z_2)$ $\mathbf{a}_{51}\ (W_{12}, U^*_{123}, U_{123})$
$\mathbf{S}_5\ (X_{12}, -Y_{12}, Z_{12})$ $\mathbf{a}_{45}\ (W_{512}, U^*_{5123}, U_{5123})$
$\mathbf{S}_4\ (X_{512}, -Y_{512}, Z_{512})$ $\mathbf{a}_{34}\ (c_3, s_3, 0)$

$Set10$ $\mathbf{S}_2\ (0,0,1)$ $\mathbf{a}_{12}\ (1,0,0)$
$\mathbf{S}_1\ (0, s_{12}, c_{12},)$ $\mathbf{a}_{51}\ (c_1, -s_1c_{12}, U_{12})$
$\mathbf{S}_5\ (X_1, -Y_1, Z_1)$ $\mathbf{a}_{45}\ (W_{51}, U^*_{512}, U_{512})$
$\mathbf{S}_4\ (X_{51}, -Y_{51}, Z_{51})$ $\mathbf{a}_{34}\ (W_{451}, U^*_{4512}, U_{4512})$
$\mathbf{S}_3\ (X_{451}, -Y_{451}, Z_{451})$ $\mathbf{a}_{23}\ (c_2, s_2, 0)$