Improving the robustness in motion planning of flexible systems through structural modification: a case study

Paolo Boscariol Dario Richiedei Iacopo Tamellin Alberto Trevisani

DTG, Università degli Studi di Padova, Italy

Abstract

The method proposed in this work is aimed at the residual vibration suppression for uncertain flexible systems. The approach, which is focused on enhancing robustness to unmeasured parametric deviations, combines a motion design approach with a structural modification approach. Both approaches are tested with and without the explicit inclusion of robustness constraints. The application to a numerical test case is provided to highlight the advantages brought by the concurrent approach. The results show that even a minimal alteration to the motion profile and to the physical properties of the plant to be moved can significantly reduce the sensitivity to large mismatches between the assumed and the actual plant.

1 Introduction

Ensuring high speed motion with minimal vibration is still an open challenge for both mechatronics practitioners and researchers. The lightweight design of robots and automatic machines is aimed at reducing inertia to boost operative speed, at the price of more pronounced oscillatory phenomena. When improperly controlled, vibration can cause all sorts of problems, including amplified mechanical stresses and reduced precision in the execution of a given motion. In many cases the motion-induced oscillation can also extend after motion completion: as such, they are usually referred to as 'residual oscillations'. Residual oscillations are commonly found in underactuated systems. While oscillations that happen during motion can be effectively damped by using one of the countless control strategies developed for this purpose [1, 5, 8], residual vibration suppression can be obtained by a careful design of the reference input to the control [16]. A wide range of solutions have been proposed with this aim, however reviewing them is outside of the scope of this work. Such a wide variety of solutions includes many methods, which can be ranked in terms of their complexity. On the higher end of the spectrum, both in terms of complexity and effectiveness, several model-based methods can be found [2]. Among them, many make use of calculus of variation to set-up a motion design problem as an optimal control one, by defining and solving a TPBP (Two-Point Boundary Value Problem) [4, 15].

On the opposite side of the spectrum one can find several model-free approaches, in which, mainly, the reduction of motion-induced vibrations is deputed to the enhancement of the smoothness of the motion profile [3, 7]. Generally these approaches are less effective that the model-based ones, but they apply to a larger variety of situations. Their applicability to all cases in which a dynamic model of the plant to be dealt with is unavailable a clear advantage for the practitioner over model-based methods. Within a 'middle ground', methods based on input shaping and filtering techniques can be found. These methods can be classified as 'loosely model-based' techniques, as they generally require just a minimal representation of the plant, typically by means of basic data such as natural frequencies and damping factors. The most relevant and popular example is the input shaping method, in the form of the Zero Vibration (ZV) shaper. or in the form of one of its many

This is a pre-print of the article: P. Boscariol, D. Richiedei, I. Tamellin, A. Trevisani
Improving the robustness in motion planning of flexible systems through structural modification: a case study
proceedings of the IEEE 17th International Conference on Advanced Motion Control, February 18-20, 2022 Padova, Italy
http://dx.doi.org/10.1109/AMC51637.2022.9729323

variations [11]. Shapers allow to alter any motion profile allowing for nominal zero residual vibration. The actual residual vibration suppression is however limited by the modeling accuracy, which justifies the development and use of robust shapers [12,14], which retain an effective residual vibration reduction within larger bounds of uncertainty.

Facing the robustness issue in motion design is the topic of this work, which proposes to augment a well-established approach, based on shaping techniques, with an approach based on structural modification. It is well known that the behaviour of a system is determined by its eigenstructure, hence it is possible to 'shape' or 'enhance' the behaviour of a system by manipulating its eigenvalues and eigenvectors. Commonly these techniques are used to tune natural frequencies [13], antiresonances [9], and damping elements [10], just to cite a few examples.

Here a structural modification approach is proposed with the specific intent of enhancing the parametric robustness of a system. The test case of choice, i.e. a pendulum with four degrees of freedom moved by the linear displacement of a cart, is used to show how the robustness to unmodeled physical alterations can be tackled, both separately and simultaneously, by acting on the design of the motion profile as well as by acting on the physical alteration of the plant.

2 Residual vibration suppression

2.1 Input shaping

As mentioned above, a simple and rather effective technique for residual vibration suppression is based on input shaping. Shaping techniques are based on a simple concept: the cancellation of two oscillations with equal amplitude and opposite phases. Let us consider a generic rest-to-rest motion profile $y^r(t)$, which is expected to produce some residual vibrations when fed to a system that is assumed to be, without loss of generality, a single DOF oscillatory system. One way to null residual vibrations is to convolve the original, henceforth 'unshaped', motion profile y^r with two properly timed pulses of proper amplitudes: the resulting shaped reference will be referred to as $y^{r,s}$. Zero residual vibration can, in this way, be obtained for any reasonable choice of y^r. The procedure just outlined refer to the popular ZV shaper, which is defined by setting the amplitudes of the two pulses as:

$$A_1 = \frac{1}{1+k}; \qquad A_2 = \frac{K}{K+1}; \tag{1}$$

with $k = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$ and with a time delay between the two pulses equal to $\tau_d/2$, i.e. half the damped oscillation period sported by the plant. The ZV shaper can be conveniently used whenever the estimated oscillating frequency $\omega_d = 2\pi/\tau_d$ and damping factor ξ reflect the actual plant, but in the presence of relevant model-plant mismatches, a residual oscillation with an amplitude proportional to the modeling error appears.

Whenever such error is too large to be tolerated, other shapers can be used: the most common solution is represented by the ZVD (Zero Vibration and Derivative) shaper. The latter is defined by enforcing that the sensitivity of the residual oscillation amplitude, V, evaluated with reference to the uncertain natural frequency w_d, i.e. $\partial V/\partial w_d$, is null. The ZVD shaper requires three pulses [14], each on separated from the previous one by $\tau_d/2$, with amplitudes:

$$A_1 = \frac{1}{1+2K+K^2}; \qquad A_2 = \frac{2K}{1+2K+K^2};$$
$$A_3 = \frac{K^2}{1+2K+K^2}; \tag{2}$$

It is worth noticing that the increased robustness comes at a cost: the delay introduced by the ZVD shaper (τ_d) is twice the one of the ZV shaper ($\tau_d/2$). This feature is relevant in all cases in which motion time is comparable with τ_d, since often the unshaped reference profile y^r is 'shortened' to obtain a specified duration of the shaped signal $y^{r,s}$. Shapers are formulated for a single oscillatory frequency, but if the plant has several frequencies to be damped, two or more shapers can be simple cascaded.

2.2 Structural modification

In this section a brief outline of the proposed procedure for the structural optimization, with and without robustness constraints, is given. The reader can refer to the recent work [6] for a more detailed explanation. Let us assume that the system whose properties are to be assigned by the structural modification can be described as a N-DOFs undamped system by the equation:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{B}\mathbf{f}(t) \tag{3}$$

Matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{B}$ are, respectively, the mass matrix, the stiffness matrix and the force distribution matrix. The external forces are collected in vector $\mathbf{f}$. Equation (3) can be re-written by switching to the frequency domain as:

$$\left(-\omega_{n,i}^2\mathbf{M} + \mathbf{K}\right)\mathbf{u}_i = 0 \qquad \text{with } i = 1, \ldots, N \tag{4}$$

in which appears the eigenpair $(\omega_{n,i}, \mathbf{u}_i)$, as composed by the i-th natural frequency $\omega_{n,i}$ and the i-th mode shape $\mathbf{u}_i$. The formulation in (4) is useful to set-up an inverse dynamic structural modification problem, which identifies the modifications $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$ to matrices $\mathbf{M}$ and $\mathbf{K}$ needed to alter the i-th eigenpair $(\omega_{n,i}, \mathbf{u}_i)$ to $(\bar{\omega}_{n,i}, \bar{\mathbf{u}}_\mathbf{i})$. If just n_d of the N eigenpairs are requested to assume some specific values, the problem is referred to as a partial structural modification problem, which can be defined, analytically, as:

$$\left(-\bar{\omega}_{n,i}^2\left(\mathbf{M} + \mathbf{\Delta M}\right) + \mathbf{K} + \mathbf{\Delta K}\right)\bar{\mathbf{u}}_i = 0 \quad \text{with} \quad i = 1, ..., n_d \tag{5}$$

The solution of (5) is in general not trivial, so a common approach is to replace it with the less challenging solution of the minimization problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{i=1}^{n_d} \left\| \left(-\bar{\omega}_{n,i}^2\left(\mathbf{M} + \mathbf{\Delta M}(\mathbf{x})\right) + \mathbf{K} + \mathbf{\Delta K}(\mathbf{x})\right)\bar{\mathbf{u}}_i \right\|_2^2 \\ \text{s.t.} \quad & \mathbf{x}^L \leq \mathbf{x} \leq \mathbf{x}^U \quad \cup \quad \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned} \tag{6}$$

in which the set of parameter changes introduced to obtain the desired eigenpairs is collected in vector $\mathbf{x}$. Such changes are assumed to be upper and lower bounded, and subjected to some used-defined linear constraints. The problem set in (5) and re-cast in (6) is a 'nominal' one, as it does not take into account the possibility of any modeling discrepancy between the theoretical model (as in (3)) and the actual one. Modeling errors and unmeasured alterations can significantly affect the effectiveness of the modifications prescribed by solving (6). Hence, a robust counterpart of the outlined procedure is needed. Sensitivity functions are used, in the proposed method, to define a robust partial assignment problem.

Sensitivity functions are the analytic representation of how the dynamics of a system is affected by the change in value of one (or more) parameter [4]: in particular here they are used to quantify the change in the the desired i-th natural frequency as a result of the change of the parameter η_j, hence they are defined by the partial derivative:

$$\bar{S}_{i,j} = \frac{\partial \bar{\omega}_{n,i}^2}{\partial \eta_j} = \frac{\bar{\mathbf{u}}_i^T\left(-\bar{\omega}_{n,i}^2\mathbf{J}_{\mathbf{M}+\mathbf{\Delta M}}^{\eta_\mathbf{j}} + \mathbf{J}_{\mathbf{K}+\mathbf{\Delta K}}^{\eta_\mathbf{j}}\right)\bar{\mathbf{u}}_i}{\bar{\mathbf{u}}_i^T\left(\mathbf{M}(\boldsymbol{\eta}) + \mathbf{\Delta M}(\mathbf{x}, \boldsymbol{\eta})\right)\bar{\mathbf{u}}_i} \tag{7}$$

where the Jacobian matrices for the modified system are:

$$\begin{aligned} \mathbf{J}_{\mathbf{M}+\mathbf{\Delta M}}^{\eta_\mathbf{j}} &= \frac{\partial\left(\mathbf{M}(\boldsymbol{\eta}) + \mathbf{\Delta M}(\mathbf{x}, \boldsymbol{\eta})\right)}{\partial \eta_j} \\ \mathbf{J}_{\mathbf{K}+\mathbf{\Delta K}}^{\eta_\mathbf{j}} &= \frac{\partial\left(\mathbf{K}(\boldsymbol{\eta}) + \mathbf{\Delta K}(\mathbf{x}, \boldsymbol{\eta})\right)}{\partial \eta_j} \end{aligned} \tag{8}$$

The sensitivity function in (7) appears as $\bar{S}_{i,j}$ since such values can be set to a specific desired value through the following equation:

$$\bar{\Phi}_{i,j}(\mathbf{x}, \boldsymbol{\eta}, \bar{\mathbf{u}}_i) = 0 \tag{9}$$

where:

$$\begin{aligned} N_{i,j}(\mathbf{x}, \boldsymbol{\eta}, \bar{\mathbf{u}}_i) &= \bar{\mathbf{u}}_i^T\left(-\bar{\omega}_{n,i}^2\mathbf{J}_{\mathbf{M}+\mathbf{\Delta M}}^{\eta_\mathbf{j}} + \mathbf{J}_{\mathbf{K}+\mathbf{\Delta K}}^{\eta_\mathbf{j}}\right)\bar{\mathbf{u}}_i \\ D_{i,j}(\mathbf{x}, \boldsymbol{\eta}, \bar{\mathbf{u}}_i) &= \bar{\mathbf{u}}_i^T\left(\mathbf{M}(\boldsymbol{\eta}) + \mathbf{\Delta M}(\mathbf{x}, \boldsymbol{\eta})\right)\bar{\mathbf{u}}_i \\ \bar{\Phi}_{i,j}(\mathbf{x}, \boldsymbol{\eta}, \bar{\mathbf{u}}_i) &= D_{i,j}(\mathbf{x}, \boldsymbol{\eta}, \bar{\mathbf{u}}_i)\bar{S}_{i,j} - N_{i,j}(\mathbf{x}, \boldsymbol{\eta}, \bar{\mathbf{u}}_i) \end{aligned} \tag{10}$$

Adding the constraint in (7) to (6) allows the definition, by means of a constrained least square minimization problem, of the robust partial optimization problem as:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{i=1}^{n_d} \left\| \left(-\bar{\omega}_{n,i}^2 \left(\mathbf{M} + \mathbf{\Delta M}(\boldsymbol{\eta}, \mathbf{x})\right) + \mathbf{K} + \mathbf{\Delta K}(\boldsymbol{\eta}, \mathbf{x})\right) \bar{\mathbf{u}}_\mathbf{i} \right\|_2^2 \\ \text{s.t.} \quad & \mathbf{x}^L \leq \mathbf{x} \leq \mathbf{x}^U \quad \cup \quad \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{u}_\mathbf{i}^L \leq \mathbf{u}_\mathbf{i} \leq \mathbf{u}_\mathbf{i}^U \\ & \bar{\mathbf{\Phi}}(\mathbf{x}, \boldsymbol{\eta}, \bar{\mathbf{u}}) = 0 \end{aligned} \tag{11}$$

Solving (11) allows defining a new system that is hence modified by altering the parameters collected in $\mathbf{x}$ to obtain that not only some natural frequencies are set to specific values, but also by ensuring that the desired frequencies are assigned with prescribed levels of robustness. In this way any random alteration of the parameter η_j does impact minimally $\bar{\omega}_{n,i}$.

This procedure, which will be applied to a test case in the following section, provides a further tool minimize the residual vibration for a system characterized by underactuation, undamped oscillations and uncertainty.

3 Performance evaluation

3.1 Plant model

The system used to test, numerically, the effectiveness of the proposed model comprises a 4-DOF pendulum excited by the motion of a linearly displacing cart, as shown in Fig. 1. The latter also specifies the vector of the four independent coordinates $\mathbf{q} = [\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4]^T$ used to model the angular displacement of each mass of the pendulum. The input of the dynamic model of the system is assumed to be the cart acceleration $\ddot{y}$. The model system dynamics, written assuming small angular displacements, can be written as:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{B}\ddot{y}(t) \tag{12}$$

with mass matrix $\mathbf{M}$, stiffness matrix $\mathbf{K}$ and force distribution vector $\mathbf{B}$ defined as:

$$\mathbf{M} = \begin{bmatrix} m_1L_1 & m_2L_2 & m_3L_3 & m_4L_4 \\ & m_2L_2 & m_3L_3 & m_4L_4 \\ & & m_3L_3 & m_4L_4 \\ & & & m_4L_4 \end{bmatrix} \begin{bmatrix} L_1 & & & \\ L_1 & L_2 & & \\ L_1 & L_2 & L_3 & \\ L_1 & L_2 & L_3 & L_4 \end{bmatrix} \tag{13}$$

$$\mathbf{K} = g \begin{bmatrix} m_1 + m_2 + m_3 + m_4 & & & \\ & m_2 + m_3 + m_4 & & \\ & & m_3 + m_4 & \\ & & & m_4 \end{bmatrix} \begin{bmatrix} L_1 & & & \\ & L_2 & & \\ & & L_3 & \\ & & & L_4 \end{bmatrix} \tag{14}$$

$$\mathbf{B} = -\begin{bmatrix} (m_1 + m_2 + m_3 + m_4)L_1 \\ (m_2 + m_3 + m_4)L_2 \\ (m_3 + m_4)L_3 \\ m_4L_4 \end{bmatrix} \tag{15}$$

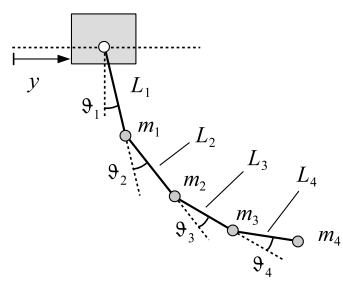

Figure 1: System model

This model is formally simple, and its matrix formulation has a clear regular formulation, however an analytic representation of the parameters relevant to this works, i.e. the natural frequencies ω_1, ω_2, ω_3 and ω_4 is rather complex and hardly usable fo any analytic manipulation. Table 1 shows the system parameters, referring to the 'original' system, i.e. the one defined prior to any structural modification.

3.2 Robust structural modification

As mentioned before, the testbench pendulum eigenstructure can be modified by means of the alteration

Table 1: Original and modified system parameters

Parameter	Value Original	Value Modified	Unit
m_1	0.1	0.206	kg
m_2	0.1	0.4	kg
m_3	0.1	0.165	kg
m_4	0.1	0.1	kg
L_1	0.35	0.7	m
L_2	0.35	0.308	m
L_3	0.35	0.255	m
L_4	0.35	0.136	m
$\omega_{n,1}$	3.01	3.01	rad/s
$\omega_{n,2}$	6.99	7.01	rad/s
$\omega_{n,3}$	11.37	12.18	rad/s
$\omega_{n,4}$	16.22	14.49	rad/s

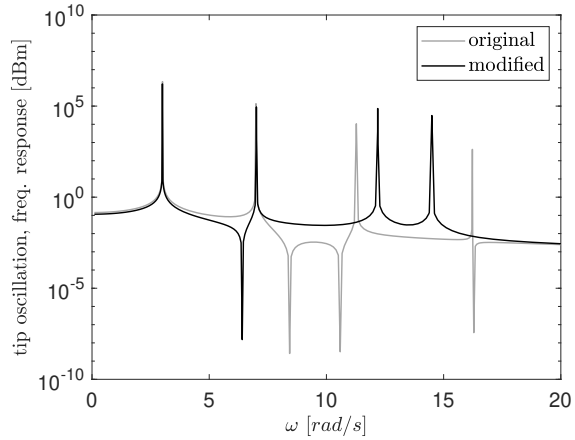

Figure 2: Frequency response of the tip mass oscillation - original and modified system

of its main parameters: here the goal is set to reduce the sensitivity $\partial\omega_{n,1}^2/\partial m_4$ by 50%. This structural modification is performed by solving the optimization problem in (11): the vector of the tuning variables $\mathbf{x}$ includes the four lengths L_1, L_2, L_3 and L_4, and the weight of the masses m_1, m_2 and m_3. The value of m_4 is not included in $\mathbf{x}$ as it is the uncertain parameter. Some constraints are included in the actual implementation of (11) in order to provide a fair comparison between the 'original' and the 'modified' system: the overall length of the pendulum is fixed, and the length of each cable section is bounded to be within 0.07 m and 0.7 m. It is also assumed that the weight of each mass can only be increased, up to 0.4 kg. The assignment of the natural frequencies is partial, so that both pendulums share the same value of the first natural frequency, i.e. $\omega_{n,1} = 3.01\ rad/s$.

The parameters of the modified system are listed in Table 1, which shows that the first natural frequency is kept unchanged by the application of the structural modification. Although not reported, the goal of halving the sensitivity $\partial\omega_{n,1}^2/\partial m_4$ is met with success, as the actual reduction is equal to 52%.

The frequency response of the systems, evaluated using as input the cart acceleration and as the output the oscillation of mass m_4, with and without the inclusion of the structural modification that enhances the robustness, are shown in fig. 2.

3.3 Robustness evaluation

The top graph shown in Fig. 3 shows the tip oscillation when the cart is moved according to a generic (i.e. 'unshaped') rest-to-rest motion profile, here defined as a quintic polynomial law with null initial and final velocities and accelerations: large residual vibrations appear after motion completion. Motion time is set to 4 s and the overall cart displacement is set to 0.6 m. The bottom graph shows instead the effects of applying the same motion profile, after filtering it with four cascaded ZV shapers, to the nominal pendulum with and without the alteration, by +20%, of mass m_4.

The results of this simple test highlight that the use of ZV shapers leads to zero residual vibration only if the natural frequency can be precisely predicted, which does not happen whenever a single parameter, such a mass m_4 is altered form its nominal value.

The effects of the perturbation on the zero residual vibration goal attainment are usually captured by sensitivity graphs, which plot the residual peak-to-peak vibration amplitude vs the value of the uncertain parameter, which is here (and for the rest of

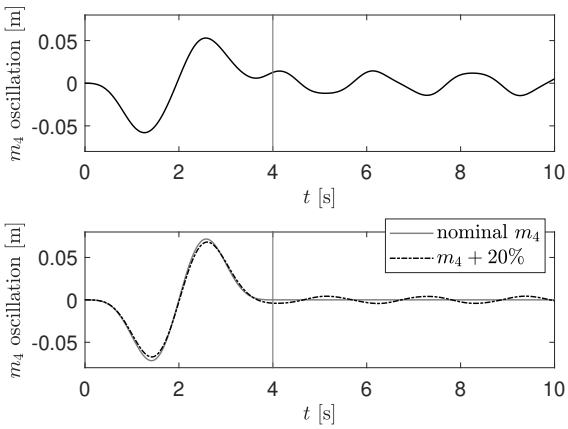

Figure 3: Top graph: resulting vibration for an unshaped motion profile. Bottom graph: oscillation for a shaped motion profile, for the nominal and a perturbed system ($m_4 + 20\%$)

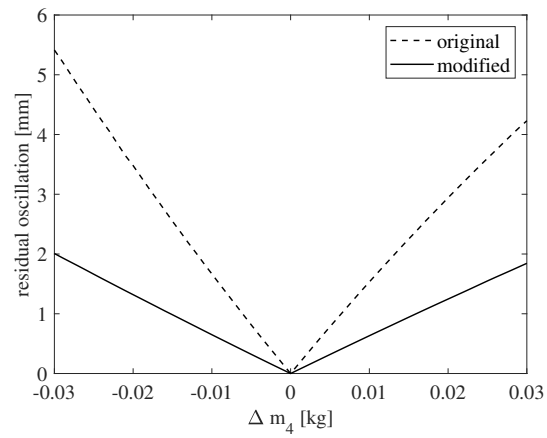

Figure 4: ZV shaper: residual vibration amplitude vs. Δm_4 - original and modified system

the paper as well), assumed to be the endpoint mass m_4. In all the tests whose results are included here, the duration of the unshaped motion profile is precompensated to keep the overall motion time, set to $T_f = 5\ s$, unaffected by the shaping action.

Here two of such graphs are included as Figures 4 and 5. The first one, shown in Fig. 4 displays the amplitude of mass m_4 oscillation vs. the perturbation of its weight (within a $\pm 30\%$ range) when using cascaded ZV shapers to filter the cart position reference signal. The dashed line refers to the original system, the solid line to the modified one. The results clearly show the sensible improvement brought from the application of the robust structural modification: the amplitude of residual vibrations, measured as the swing of mass m_4, is more than halved for any value within the whole range of perturbations considered in the test.

The improvement brought by the structural modification is even more evident when the reference signal is filtered by ZVD shapers: Fig. 5 show that the approach that combines the enforcement of robustness conditions both at shaper design level and at system design level (by the proposed structural modification procedure) is very effective. The numerical results show that residual vibrations are, in this case, kept below 0.1 mm even when m_4 is increased, or decreased, by as much as 30%.

The motion profiles used to compute the sensitivity graphs of Fig. 4 and 5 are displayed in Fig. 6, which shows the speed and acceleration profiles generated after the application of the ZV and ZVD shapers according to the natural frequencies of the original system. The motion profiles generated for the modified system are not displayed, being very similar to the ones represented in the graphs. As expected, the application of robust shapers requires, in comparison with 'standard' shapers, higher peak acceleration values and the frequency content of the acceleration signal is moved to higher frequencies. For this reason shapers are usually scarcely effective whenever the motion time is comparable to the oscillation periods of the system to be damped [4].

Conclusion

In this paper the issue of residual vibration suppression for uncertain systems is dealt with. The proposed method combines a robust approach to both motion design and structural modification, with the

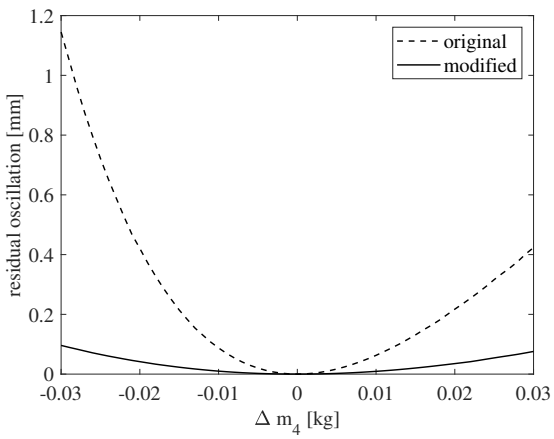

Figure 5: ZVD shaper: residual vibration amplitude vs. Δm_4 - original and modified system

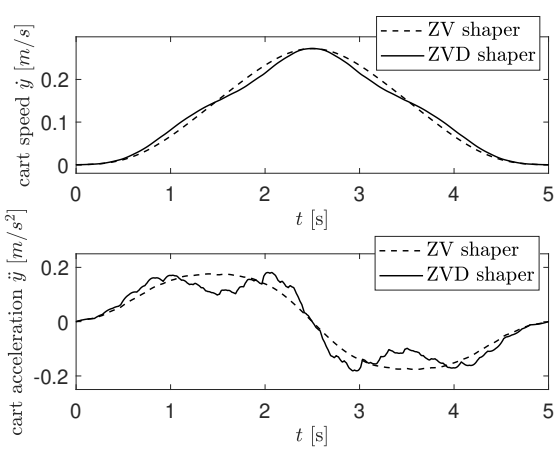

Figure 6: Speed and acceleration profiles with the application of the ZV and ZVD shapers

aim of alleviating the problem of accurately estimating the model of a system to be actuated with minimal vibration excitation. The robustification, i.e. the reduction of sensitivity to unmodeled parameter changes, can be obtained, as shown in this work, not only through robust input shaping, but also by introducing some physical alterations to the system to be moved with minimal residual vibrations. For this aim a robust inverse dynamic structural modification procedure is cast by incorporating into the design problem some constraints by means of the sensitivity functions of the system to be moved. Numerical results showcase the effectiveness of the proposed robust structural modification approach which, when combined the enhancement of the robustness obtained at the motion design level, allows achieving a relevant reduction of the residual vibration despite the presence of sizeable unmodeled and unmeasured parametric changes.

References

[1] Rabih Alkhatib and MF Golnaraghi. Active structural vibration control: a review. *Shock and Vibration Digest*, 35(5):367, 2003.

[2] ATEF A Ata. Optimal trajectory planning of manipulators: a review. *Journal of Engineering Science and technology*, 2(1):32–54, 2007.

[3] Pierre-Jean Barre, Richard Bearee, Pierre Borne, and Eric Dumetz. Influence of a jerk controlled movement law on the vibratory behaviour of high-dynamics systems. *Journal of Intelligent and Robotic Systems*, 42(3):275–293, 2005.

[4] Paolo Boscariol and Dario Richiedei. Robust point-to-point trajectory planning for nonlinear underactuated systems: Theory and experimental assessment. *Robotics and Computer-Integrated Manufacturing*, 50:256–265, 2018.

[5] Giovanni Boschetti, Dario Richiedei, and Alberto Trevisani. Delayed Reference Control Applied to Flexible Link Mechanisms: A Scheme for Effective and Stable Control. *Journal of*

Dynamic Systems, Measurement, and Control, 134(1), 12 2011. 011003.

[6] Roberto Caracciolo, Dario Richiedei, and Iacopo Tamellin. Robust assignment of natural frequencies and antiresonances in vibrating systems through dynamic structural modification. *Shock and Vibration*, 2021, 2021.

[7] A Gasparetto, A Lanzutti, R Vidoni, and V Zanotto. Experimental validation and comparative analysis of optimal time-jerk algorithms for trajectory planning. *Robotics and Computer-Integrated Manufacturing*, 28(2):164–181, 2012.

[8] Chang Tai Kiang, Andrew Spowage, and Chan Kuan Yoong. Review of control and sensor system of flexible manipulator. *Journal of Intelligent & Robotic Systems*, 77(1):187–213, 2015.

[9] Dario Richiedei, Iacopo Tamellin, and Alberto Trevisani. Beyond the tuned mass damper: a comparative study of passive approaches to vibration absorption through antiresonance assignment. *Archives of Computational Methods in Engineering*, pages 1–26, 2021.

[10] Rafael A Rojas, Erich Wehrle, and Renato Vidoni. Optimal design for the passive control of vibration based on limit cycles. *Shock and Vibration*, 2019, 2019.

[11] William Singhose, Lisa Porter, Michael Kenison, and Eric Kriikku. Effects of hoisting on the input shaping control of gantry cranes. *Control engineering practice*, 8(10):1159–1165, 2000.

[12] William Singhose and Joshua Vaughan. Reducing vibration by digital filtering and input shaping. *IEEE Transactions on Control Systems Technology*, 19(6):1410–1420, 2010.

[13] Sung-Han Tsai, Huajiang Ouyang, and Jen-Yuan Chang. Inverse structural modifications of a geared rotor-bearing system for frequency assignment using measured receptances. *Mechanical Systems and Signal Processing*, 110:59–72, 2018.

[14] Joshua Vaughan, Aika Yano, and William Singhose. Comparison of robust input shapers. *Journal of Sound and Vibration*, 315(4-5):797–815, 2008.

[15] Oskar Von Stryk and Roland Bulirsch. Direct and indirect methods for trajectory optimization. *Annals of operations research*, 37(1):357–373, 1992.

[16] Xumiao Xie, Jie Huang, and Zan Liang. Vibration reduction for flexible systems by command smoothing. *Mechanical Systems and Signal Processing*, 39(1-2):461–470, 2013.